Lecture XIV: Sheaf Theory II

Last time, we defined presheaves & sheaves.

Definition: A presheaf on a topological space X with values in objects of a category $\mathcal{C}$ is a pair $(\mathcal{F}, \rho)$ satisfying:

(1) $\mathcal{F}$ is an assignment : $U \subseteq X$ open $\longmapsto \underset{=\Gamma(U,\mathcal{F})}{\mathcal{F}(U)} \in \mathrm{Obj}(\mathcal{C})$. (sections on U)

$\mathcal{F}(\emptyset)$ is the terminal object in $\mathcal{C}$ ($\emptyset$ for $\mathcal{C}=$ set, $\{0\}$ for $G=$ AbGp, Ring, $\mathbb{K}$-v.sp.)

(2) For each pair $V \subseteq U \subseteq X$ of opens, we have $\mathcal{F}(U) \xrightarrow[\text{(restriction map)}]{\rho_{U,V}} \mathcal{F}(V) \in \mathrm{Homs}(\mathcal{C})$

satisfying the following properties:

(i) $\rho_{U,U} = \mathrm{id}_{\mathcal{F}(U)}$ $\quad \forall\, U \subseteq X$ open

(ii) For each triple $W \subseteq V \subseteq U$ of opens in X we have $\rho_{U,W} = \rho_{V,W} \circ \rho_{U,V}$

Definition: A presheaf is a sheaf if it satisfies the cocycle condition: $\forall\, U \subseteq X$ open & all open covers $\{U_i\}_{i\in I}$ of U & $f_i \in \mathcal{F}(U_i)$ with $f_{ij} = f_i|_{U_i\cap U_j} = f_j|_{U_i \cap U_j} = f_{ji}$

$\exists!\, f \in \mathcal{F}(U)$ with $f|_{U_i} = f_i \quad \forall i$

Main examples: ① Locally constant maps from X to a set Y

② Regular maps to $\mathbb{A}^1_{\mathbb{K}}$ from an irreducible variety $W \subseteq \mathbb{A}^n_{\mathbb{K}}$

Remark: We can build new sheaves from old ones via direct sums, direct products & inverse limits (see HW4)

§1 Stalks:

Fix a presheaf $\mathcal{F}$ on a topological space X.

Definition: Given $x \in X$, we define the stalk of $\mathcal{F}$ at x as a direct limit

$$\mathcal{F}_x = \varinjlim_{\substack{U \text{ open} \\ x\in U}} \mathcal{F}(U) = \bigsqcup_{x\in U\subseteq X} \mathcal{F}(U)/\sim_x \qquad \text{where } \sim \text{ is defined}$$

as follows: if $f_1 \in \mathcal{F}(U_1)$, $f_2 \in \mathcal{F}(U_2)$ we say $f_1 \sim_x f_2$ if $\exists\, V \underset{\text{open}}{\subseteq} U_1 \cap U_2$ with $x \in V$ st $f_1|_V = f_2|_V$ (This is an equivalence relation)

Note: The construction comes with a natural map
$$\mathcal{F}(U) \xrightarrow{\rho_x} \mathcal{F}_x$$
$$f \longmapsto \bar{f} = \text{equiv. class of } f.$$

It is a morphism in the category where $\mathcal{F}$ lies

__Main example:__ $\mathcal{F} = \mathcal{O}_W \rightsquigarrow \mathcal{O}_{p,W} = \mathcal{F}_p = \{\text{regular functions at } p\}/\sim$

If $\overline{\mathbb{K}} = \mathbb{K}$, then $\mathcal{O}_p = \mathcal{O}_{p,W} = \mathbb{K}[W]_{\mathfrak{m}_p}$. (Theorem §13.1)

__Lemma 1:__ Suppose $\mathcal{F}$ is a sheaf on X. Given any $U \subseteq X$ open, the map

$$\begin{aligned} \mathcal{F}(U) &\longrightarrow \prod_{x\in U} \mathcal{F}_x \\ f &\longmapsto (\rho_x(f))_x \end{aligned}$$

is injective.

__Proof:__ Pick $s, s' \in \mathcal{F}(U)$ with $\rho_x(s) = \rho_x(s')$ $\forall x \in U$. By the definition of the stalk $\mathcal{F}_p$, we can find an open V_p with $p \in V_p \subseteq V$ with

$s|_{V_p} = s'|_{V_p}$. Since $\{V_p\}_{p\in V}$ is an open cover of V & $\{s|_{V_p} \in \mathcal{F}(V_p)\}_{p\in V}$ agree on the overlaps, the sheaf axiom confirms that s is the unique element of $\mathcal{F}(U)$ restricting to $s|_{V_p}$ on each V_p. But since s' also satisfies this condition, we conclude that $s = s'$ on $\mathcal{F}(U)$. $\square$

⚠ The statement fails if $\mathcal{F}$ is a presheaf but __not__ a sheaf.

__Example:__ $X = \{p, q\}$ with the discrete topology & take $\mathcal{F}$ to be the presheaf $\mathcal{F}$ with $\mathcal{F}(X) = \{0, 1\}$, $\mathcal{F}(\{p\}) = \mathcal{F}(\{q\}) = \mathcal{F}(\emptyset) = \{0\}$ with the restrictions uniquely determined.

- Stalks: $\mathcal{F}_p = \mathcal{F}_q = \{0\}$ is a singleton in Set, so all global sections have the same stalks.
- $\mathcal{F}$ is not a presheaf: sections on $\{p\}$ & $\{q\}$ are unique & agree on the overlap, but they don't glue to a $!$ section in $\mathcal{F}(X)$: both $\{0\}$ & $\{1\} \in \mathcal{F}(X)$ restrict to $\{0\}$ on $\{p\}$ & $\{q\}$. $\square$
- Stalks of a presheaf $\mathcal{F}$ on X can be used to build a topological space $|\mathcal{F}|$ & a natural map $p : |\mathcal{F}| \longrightarrow X$ (called the "projection map")

__Definition:__ $\boxed{|\mathcal{F}|} = \bigsqcup_{x\in X} \mathcal{F}_x$ & $\boxed{p} : |\mathcal{F}| \longrightarrow X$, $\zeta \in \mathcal{F}_x \longmapsto x$ $\quad (\zeta \in \mathcal{F}_x \leftrightarrow (x, \zeta))$

with $p|_{\mathcal{F}_x} = \rho_x$

We endow $|\mathcal{F}|$ with a topology by fixing a basis:

Definition: Given $U \subseteq X$ open & $f \in \mathcal{F}(U)$ we set

$$\mathcal{N}(U, f) = \{ \rho_x(f) \in \mathcal{F}_x : x \in U \}$$

Theorem 1: $\mathcal{B} = \{ \mathcal{N}(U, f) : U \text{ open}, f \in \mathcal{F}(U) \}$ is a basis for a topology on $|\mathcal{F}|$

Furthermore, this topology makes $p : |\mathcal{F}| \longrightarrow X$ into a local homeomorphism.

Proof: Exercise (see HW4)

Q: How good is this topology?

A: It depends on the topology of X & how can we equate sections of $\mathcal{F}$ from their equivalence class on same stalk (see HW4). Since Zariski Topology is not Hausdorff, $|\mathcal{O}_W|$ will not be a nice topological space.

Remark: Given a Top space X & a presheaf $\mathcal{F}$ on X, it is usual to refer to elements of each $\mathcal{F}(U)$ ($U \subseteq X$ open) as "sections of $\mathcal{F}$ on U". The choice is made by the following fact (see HW4). By construction, a section to $p : |\mathcal{F}| \longrightarrow X$ on U is given by

$s : \mathcal{U} \longrightarrow |\mathcal{F}|$ continuous with

(section)

$$\begin{tikzcd} & {|\mathcal{F}|} \arrow[d, "p"] \\ \mathcal{U} \arrow[ur, dashed, "s"] \arrow[r, hook] \arrow[ur, phantom, "\circlearrowleft", near end, shift right=2] & X \end{tikzcd}$$

$p \circ s = inc_{\mathcal{U}}$.

So
$$\begin{aligned} \mathcal{F}(\mathcal{U}) &\xrightarrow{\ \Phi\ } \{ \mathcal{U} \xrightarrow{s} |\mathcal{F}| \text{ continuous sections to } p \} := \mathcal{G}(\mathcal{U}) \\ f &\longmapsto (s_{(x)} = \rho_x(f) \quad \forall x \in \mathcal{U}) \end{aligned}$$

If $\mathcal{F}$ is a sheaf, then $\mathcal{G}$ is a sheaf, Φ is a bijection and $\Phi(f) = s$. $\square$

§2 Morphisms of sheaves:

Definition: If $\mathcal{F}$ & $\mathcal{G}$ are presheaves on X, a morphism $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ consists of an assignment: $\varphi(U) : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$ for each open $U \subseteq X$ compatible with the restriction maps $\rho^{\mathcal{F}}$ & $\rho^{\mathcal{G}}$. That is, given two opens $V \subseteq U$ of X, the diagram

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\varphi(U)"] \arrow[d, "\rho^{\mathcal{F}}_{UV}"'] & \mathcal{G}(U) \arrow[d, "\rho^{\mathcal{G}}_{UV}"] \\
\mathcal{F}(V) \arrow[r, "\varphi(V)"] & \mathcal{G}(V)
\end{tikzcd}$$
commutes.

Remark: A morphism of presheaves is a natural transformation $\mathcal{F} \longrightarrow \mathcal{G}$. in the category $\mathcal{C}$.

Definition: An isomorphism is a morphism of presheaves which has a two-sided inverse

Lemma 2: For any $p \in X$ & any morphism $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$, we get a map.
$\varphi_p : \mathcal{F}_p \longrightarrow \mathcal{G}_p$ where $\varphi_p(\overline{f}) = \overline{\varphi_U(f)}$. Here, $\overline{f} \leftrightarrow f \in \mathcal{F}(U)$ for $p \in U$ open

Proof: If $\overline{f_1} = \overline{f_2} \in \mathcal{F}_p$ with $f_1 \in \mathcal{F}(U_1)$, $f_2 \in \mathcal{F}(U_2)$, then $f_1|_V = f_2|_V$ for some open V with $p \in V \subseteq U_1 \cap U_2$. Thus,

$$\varphi_V(f_1|_V) = \rho_{U_1V}\,\varphi_{U_1}(f_1) = \rho_{U_2V}\,\varphi_{U_2}(f_2) = \varphi_V(f_2|_V) \quad \in \mathcal{G}(V)$$

Conclude: $\overline{\varphi_{U_1}(f_1)} = \overline{\varphi_{U_2}(f_2)}$ in $\mathcal{G}_p$, so the map φ_p is well-defined. $\square$

Remark: The maps φ_p are in the same category as φ (eg Sets, Ab, Rings, $\mathbb{K}$-v.sp.)

- Another advantage of stalks: we can check for isomorphisms.

Proposition 1: Fix a topological space X & a morphism of sheaves $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ on X. Then, φ is an isomorphism if, and only if, the induced maps on stalks $\varphi_p : \mathcal{F}_p \longrightarrow \mathcal{G}_p$ are isomorphisms $\forall p \in X$.

Proof: $(\Rightarrow)$ is clear. The definition of the maps φ_p & ψ_p for $\psi = \varphi^{-1}$ yield

$$id_{\mathcal{F}_p} = (id_{\mathcal{F}})_p = (\varphi \circ \psi)_p = \varphi_p \circ \psi_p$$
$$id_{\mathcal{G}_p} = (id_{\mathcal{G}})_p = (\psi \circ \varphi)_p = \psi_p \circ \varphi_p.$$

$(\Leftarrow)$ We need to show each φ : $\mathcal{F}(U) \longrightarrow (U)$ is an isomorphism. We do this by showing φ_U is both injective & surjective.

• φ_U is injective: Fix $s, s' \in \mathcal{F}(U)$ with $\varphi_U(s) = \varphi_U(s')$. Then, for each $p \in U$, $\varphi_p(\rho_p(s)) \underset{\text{def}}{=} \rho_p(\varphi_U(s)) = \rho_p(\varphi_U(s')) = \varphi_p(\rho_p(s')) \in \mathcal{G}_p$.

Since φ_p is injective, we conclude $\rho_p(s) = \rho_p(s')$ $\forall p \in U$. Lemma 1 then yields $s = s'$ in $\mathcal{F}(U)$.

• φ_U is surjective: Given $g \in \mathcal{G}(U)$ we want to build $f \in \mathcal{F}(U)$ with $\varphi_U(f) = g$.

For each p we can build $! \, f_p \in \mathcal{F}_p$ with $\varphi_p(f_p) = g_p$.

We identify f_p with the equivalence class of $f_p \in \mathcal{F}(U_p)$ for some open U_p with $p \in U_p \subseteq U$.

• Note: $\varphi_p(f_p) = \rho_p(\varphi_{U_p}(f_p)) = \rho_p(g_{|U_p}) \in \mathcal{G}_p$. Thus, we can replace U_p by an open V_p with $p \in V_p \subseteq U_p$ where

$$\varphi_{V_p}(f_p|_{V_p}) = \varphi_{U_p}(f_p)|_{V_p} = (g|_{U_p})_{|V_p} = g|_{V_p}$$

We simplify notation & write $f_p = f_p|_{V_p} \in \mathcal{F}(V_p)$.

• By construction, $\{V_p\}_{p \in U}$ is an open covering of U. & $\varphi_{V_p}(f_p) = g|_{V_p}$ $\forall p \in U$.

To find f with $f_{|V_p} = f_p$ it's enough to show that $\{f_p \in \mathcal{F}(U_p)\}_p$ agree on the overlaps.

Pick $p, p' \in U$ with $V_p \cap V_{p'} \neq \emptyset$, then $f_p|_{V_p \cap V_{p'}}$ & $f_{p'}|_{V_p \cap V_{p'}}$ are two elements of $\mathcal{F}(V_p \cap V_{p'})$ mapping to $g|_{V_p \cap V_{p'}}$ under $\varphi_{V_p \cap V_{p'}}$. Since $\varphi_{V_p \cap V_{p'}}$ is injective, we conclude that $f_p|_{V_p \cap V_{p'}} = f_{p'}|_{V_p \cap V_{p'}}$.

• By the sheaf axiom applied to $\{f_p \in \mathcal{F}(V_p)\}_p$ we build a unique $f \in \mathcal{F}(U)$ with $f_{|V_p} = f_p$

• To finish, we need to check $\varphi_U(f) = g$.

But this follows from the fact that

$$\rho_p(\varphi_U(f)) = \varphi_p(\rho_p(f)) = \varphi_p(\rho_p(f_p)) = g_p = \rho_p(g).$$

So $\varphi_U(f)$ & g agree on all stalks of $\mathcal{G}$, thus they give the same section on $\mathcal{G}(U)$ by Lemma 1, ie $\varphi_U(f) = g$. $\square$

<u>Remark:</u> We used the sheaf axiom several times in the proof because we invoked Lemma 1. The statement fails for presheaves (Eg X locally connected top space $\mathcal{F}$ = constant presheaf, $\mathcal{G}$ = locally constant sheaf & $\varphi : \mathcal{F} \hookrightarrow \mathcal{G}$ the natural inclusion).