Lecture XV: Sheaf Theory III

Recall Fix a presheaf $\mathcal{F}$ on a topological space X.

Definition: Given $x\in X$, we define the stalk of $\mathcal{F}$ at x as a direct limit

$$\mathcal{F}_x = \varinjlim_{\substack{U \text{ open}\\ x\in U}} \mathcal{F}(U) = \bigsqcup_{x\in U\subseteq X} \mathcal{F}(U)/\sim_x$$

where $\sim$ is defined as follows: if $f_1\in\mathcal{F}(U_1)$, $f_2\in\mathcal{F}(U_2)$ we say $f_1 \sim_x f_2$ if $\exists\ V \underset{\text{open}}{\subseteq} U_1\cap U_2$ with $x\in V$ st $f_1|_V = f_2|_V$ (This is an equivalence relation)

Note: The construction comes with a natural map
$$\mathcal{F}(U) \xrightarrow{\rho_x} \mathcal{F}_x$$
$$f \longmapsto \bar{f} = \text{equiv class of } f.$$
It is a morphism in the category where $\mathcal{F}$ lies

Main example: $\mathcal{F} = \mathcal{O}_W \rightsquigarrow \mathcal{O}_{p,W} = \mathcal{F}_p = \{\text{regular functions at } p\}/\sim$

If $\overline{\mathbb{K}} = \mathbb{K}$, then $\mathcal{O}_p = \mathcal{O}_{p,W} = \mathbb{K}[W]_{m_p}$. (Theorem §13.1)

Lemma: Suppose $\mathcal{F}$ is a sheaf on X. Given any $U\subseteq X$ open, the map
$$\mathcal{F}(U) \longrightarrow \prod_{x\in U} \mathcal{F}_x \quad \text{is injective.}$$
$$f \longmapsto (\rho_x(f))_x$$

⚠ The result fails for presheaves that are not sheaves

Example: $X = \{p,q\}$ with the discrete topology & take $\mathcal{F}$ to be the presheaf $\mathcal{F}$ with $\mathcal{F}(X) = \{0,1\}$, $\mathcal{F}(\{p\}) = \mathcal{F}(\{q\}) = \mathcal{F}(\emptyset) = \{0\}$ with the restrictions uniquely determined.

- Stalks: $\mathcal{F}_p = \mathcal{F}_q = \{0\}$ is a singleton in Set, so all global sections have the same stalks.
- $\mathcal{F}$ is not a presheaf: Sections on $\{p\}$ & $\{q\}$ are unique & agree on the overlap, but they don't glue to a ! section in $\mathcal{F}(X)$: both $\{0\}$ & $\{1\} \in \mathcal{F}(X)$ restrict to $\{0\}$ on $\{p\}$ & $\{q\}$. □

Proposition: Fix a topological space X & a morphism of sheaves (= natural transf) $\varphi: \mathcal{F} \to \mathcal{G}$ on X. Then, φ is an isomorphism if, and only if, the induced maps on stalks $\varphi_p: \mathcal{F}_p \to \mathcal{G}_p$ are isomorphisms $\forall p\in X$.

Remark: We used the sheaf axiom several times in the proof. The statement fails for presheaves (Eg $\mathcal{F}$ = constant presheaf, $\mathcal{G}$ = locally constant sheaf & $\varphi: \mathcal{F} \hookrightarrow \mathcal{G}$ the natural inclusion).

§1. Sheafification:

• Next, we construct sheaves from presheaves via universal property:

Proposition 1: Given a presheaf $\mathcal{F}$, there is a sheaf $\mathcal{F}^a$ & a morphism $\theta: \mathcal{F} \to \mathcal{F}^a$ satisfying the following universal property: For each sheaf $\mathcal{G}$ & each morphism $\varphi: \mathcal{F} \longrightarrow \mathcal{G}$, $\exists !\ \psi: \mathcal{F}^a \longrightarrow \mathcal{G}$ with $\varphi = \psi \circ \theta$.

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\varphi"] \arrow[d, "\theta"'] & \mathcal{G} \\ \mathcal{F}^a \arrow[ur, dashed, "\psi"'] & \end{tikzcd}$$

Furthermore, the pair $(\mathcal{F}^a, \theta)$ is unique up to unique isomorphism.

Definition: We call $\mathcal{F}^a$ the sheaf associated to the presheaf $\mathcal{F}$.

Remark: By construction $\mathcal{F}^a_p = \mathcal{F}_p$ $\forall p \in X$, & $(\mathcal{F}^a, \theta) = (\mathcal{F}, \mathrm{id}_{\mathcal{F}})$ whenever $\mathcal{F}$ is a sheaf.

Proof of Proposition 1: • Uniqueness follows by universal property: Take $\mathcal{G} = \tilde{\mathcal{F}}^a$, $\varphi = \tilde{\theta}$ to conclude that ψ is invertible & $\psi^{-1} = \tilde{\psi}$.

Uniqueness of ψ & $\tilde{\psi}$ gives:

$$\begin{tikzcd} \mathcal{F} \arrow[rr, "\theta"] \arrow[d, "\theta"'] \arrow[dr, "\tilde{\theta}"] & & \mathcal{F}^a \\ \mathcal{F}^a \arrow[r, dashed, "\psi"'] \arrow[urr, bend right=60, "\mathrm{id}_{\mathcal{F}^a}"'] & \tilde{\mathcal{F}}^a \arrow[ur, dashed, "\tilde{\psi}"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{F} \arrow[rr, "\tilde{\theta}"] \arrow[d, "\tilde{\theta}"'] \arrow[dr, "\theta"] & & \tilde{\mathcal{F}}^a \\ \tilde{\mathcal{F}}^a \arrow[r, dashed, "\tilde{\psi}"'] \arrow[urr, bend right=60, "\mathrm{id}_{\tilde{\mathcal{F}}^a}"'] & \mathcal{F}^a \arrow[ur, dashed, "\psi"'] & \end{tikzcd}$$

• Existence: We need to ensure the cocycle condition on $\mathcal{F}^a$ & the fact that $\mathcal{F}^a_p = \mathcal{F}_p$ $\forall p$.

Given $U \subseteq X$ open we define

$$\mathcal{F}^a(U) := \Big\{ s: U \longrightarrow |\mathcal{F}_{|U}| = \bigsqcup_{p \in U} \mathcal{F}_p \;\Big|\; \overbrace{(1): s(p) \in \mathcal{F}_p}^{s \text{ is a section to } p: |\mathcal{F}_{|U}| \to U} \ \& \ (2): \forall p \in U\ \exists\ V \text{ open with } p \in V \subseteq U \ \& \ t \in \mathcal{F}(V) \text{ st } \rho_q(t) = s(q)\ \forall q \in V \Big\}$$

Condition (2) ensures agreement on overlaps & cocycle condition.

- Take the restriction map on $\mathcal{F}^a$ to be the natural one: $\rho_{UW}(s) = s|_W$. $\forall\ W \subseteq U$ opens.
- $\theta : \mathcal{F} \to \mathcal{F}^a$ via $\theta_U : \mathcal{F}(U) \longrightarrow \mathcal{F}^a(U)$, $f \longmapsto \{ p \overset{s}{\longmapsto} \rho_p(f) \}$ (s lifts to f so (2) holds)
- Check: $\mathcal{F}$ is a sheaf & it satisfies the universal property □

Example. $\mathcal{F}$ = constant maps $\rightsquigarrow$ $\mathcal{F}^a$ = locally constant maps.

§2. Exact sequences:

Fix $\mathcal{C}$ = Ab, $\mathbb{K}$-vsp., Mod(R) (R comm ring) or $\mathcal{O}_X$-Mod for $(X, \mathcal{O}_X)$ a ringed space (eg X an affine variety) — top space, sheaf of comm. rings

For $\mathcal{O}_X$-Mod: $U \longmapsto \mathcal{F}(U) \in \text{Mod}(\mathcal{O}_X(U))$ [Aside: for locally ringed space $\mathcal{O}_{X,p}$ must be local $\forall p$]

KEY: These choices are abelian categories (have kernels & cokernels)

Definition: A subsheaf of a sheaf $\mathcal{F}$ is a sheaf $\mathcal{F}'$ s.t $\forall\ U \subseteq X$ open, $\mathcal{F}'(U)$ is a subobject of $\mathcal{F}(U)$ in $\mathcal{C}$ & the restriction maps ρ'_{UV} are induced by ρ_{UV}.

Remark: We have $\mathcal{F}'_p$ is a subobject of $\mathcal{F}_p$ $\forall p \in X$.

Example: $\mathcal{F} = \mathcal{O}_W$ for W irreducible affine variety & $\mathcal{F}'$ = locally constant maps to $\mathbb{A}^1_{\mathbb{K}}$.

Definition: Fix $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ a morphism of presheaves on a topological space X with values in $\mathcal{C}$. We define the presheaves kernel of φ, image of φ & cokernel of φ via the assignments:

$\ker_U : U \longmapsto \ker(\varphi_U) \subset \mathcal{F}(U)$, $\text{image}_U : U \longmapsto \text{image}(\varphi_U) \subseteq \mathcal{G}(U)$ &

$\text{cokernel}_U : U \longmapsto \text{cokernel}(\varphi_U) \subseteq \mathcal{G}(U) / \text{image}(\varphi_U)$ for each open U of X.

Lemma 1: If φ is a morphism of sheaves, then ker is a sheaf. However cokernel & image are almost never a sheaf.

Proof: Exercise.

Remark: $\ker_U$ is a subsheaf of $\mathcal{F}$ whenever $\mathcal{F}$ is a sheaf on X

Definition: $\operatorname{im}\varphi = (\operatorname{image}\varphi)^a$ & $\operatorname{coker}\varphi = (\operatorname{cokernel}\varphi)^a$

Remark: $\operatorname{im}\varphi$ comes with an injective map $\operatorname{im}\varphi \xrightarrow{\psi} \mathcal{G}$ (by

$$\begin{tikzcd} \operatorname{image}\varphi \arrow[r, "inc"] \arrow[d, "\theta"'] & \mathcal{G} \\ \operatorname{im}\varphi \arrow[ur, dotted, "\exists!\psi"'] & \end{tikzcd}$$

)

so $\operatorname{im}\varphi$ is a subsheaf of $\mathcal{G}$.

Definition: A morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ is

(1) injective if $\ker\varphi = 0$ ($\Leftrightarrow \forall U \subseteq X$ open $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is inj)

(2) surjective if $\operatorname{im}\varphi = \mathcal{G}$

⚠ φ surjective $\not\Rightarrow$ all φ_U's are surjective. This is so because $\operatorname{im}\varphi$ requires sheafification. The surjectivity condition can be checked on stalks! (see HW 4)

Definition: A sequence of morphisms of sheaves

$$(*) \quad \cdots \longrightarrow \mathcal{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathcal{F}^{i} \xrightarrow{\varphi^{i}} \mathcal{F}^{i+1} \longrightarrow \cdots$$

is exact if $\ker\varphi^i = \operatorname{im}\varphi^{i-1}$ $\forall i$.

In particular: (1) $0 \longrightarrow \mathcal{F} \xrightarrow{\varphi} \mathcal{G}$ is exact if, and only if, φ is injective.

(2) $\mathcal{F} \xrightarrow{\varphi} \mathcal{G} \to 0$ is exact if, and only if, φ is surjective

Proposition 3: The sequence $(*)$ is exact $\Leftrightarrow$ the associated sequence of stalks $\cdots \to \mathcal{F}_p^{i-1} \xrightarrow{\varphi_p^{i-1}} \mathcal{F}_p^{i} \xrightarrow{\varphi_p^{i}} \mathcal{F}_p^{i+1} \to \cdots$ is exact.

Proof: Injective & surjective can be checked on stalks + Sheafification preserves stalks.

Definition: Fix $\mathcal{F}'$ a subsheaf of a sheaf $\mathcal{F}$. The quotient sheaf $\mathcal{F}/\mathcal{F}'$ is defined as the sheafification of the presheaf $U \longmapsto \mathcal{F}(U)/\mathcal{F}'(U)$

The cokernel of a morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ is the quotient sheaf $\mathcal{G}/\operatorname{im}\varphi$

(We factor $\mathcal{F} \xrightarrow{\varphi} \mathcal{G}$ via $\mathcal{F} \longrightarrow \operatorname{im}\varphi \hookrightarrow \mathcal{G}$)

§ 3 Sheaf Cohomology:

As in §2, fix $\mathcal{C} =$ Ab, $\mathbb{K}$-v.sp, Mod(R) for R comm. ring, $\mathcal{O}_X$-mod

$Sh(X)$ = category of sheaves on X with values in $\mathcal{C}$ (Objects = sheaves, Maps = morphisms of sheaves)

- Consider the global sections functor Γ:

$$\begin{array}{ccc} Sh(X) & \xrightarrow{\Gamma} & \mathcal{C} \\ \mathcal{F} & \longmapsto & \Gamma(X,\mathcal{F}) := \mathcal{F}(X) \end{array}$$

- Γ is an additive, left-exact covariant functor.

(Additive: $\mathcal{F} \oplus \mathcal{G} \xrightarrow{\Gamma} \Gamma(\mathcal{F}) \oplus \Gamma(\mathcal{G})$ "Γ respects $\oplus$")

(Left-exact: $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ ses $\Rightarrow 0 \to \Gamma(X,\mathcal{F}_1) \to \Gamma(X,\mathcal{F}_2) \to \Gamma(X,\mathcal{F}_3)$ is exact)

- Sheaf cohomology is built out of the right-derived functors of Γ

$$R^i\Gamma : Sh(X) \to \mathcal{C} \quad \forall i \geq 0$$

These functors are characterize by the following properties:

(1) $R^0\Gamma = \Gamma$

(2) For every short exact sequence $0 \to \mathcal{F}_1 \xrightarrow{\varphi} \mathcal{F}_2 \xrightarrow{\psi} \mathcal{F}_3 \to 0$ in $Sh(X)$ we get a long exact sequence in $\mathcal{C}$:

$$\begin{array}{l} 0 \longrightarrow \mathcal{F}_1(X) \xrightarrow{\varphi_X} \mathcal{F}_2(X) \xrightarrow{\psi_X} \mathcal{F}_3(X) \\ \xrightarrow{\delta^1} R^1\Gamma(\mathcal{F}_1) \longrightarrow R^1\Gamma(\mathcal{F}_2) \longrightarrow R^1\Gamma(\mathcal{F}_3) \\ \xrightarrow{\delta^2} R^2\Gamma(\mathcal{F}_2) \longrightarrow \cdots \end{array}$$

δ^i = connecting morphism

(3) [Functoriality] Given a morphism between 2 s.e.s in $Sh(X)$

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{F}_1 & \to & \mathcal{F}_2 & \to & \mathcal{F}_3 & \to & 0 \\ & & \downarrow \phi_1 & & \downarrow \phi_2 & & \downarrow \phi_3 & & \\ 0 & \to & \mathcal{G}_1 & \to & \mathcal{G}_2 & \to & \mathcal{G}_3 & \to & 0 \end{array}$$

we have

$$\begin{array}{ccc} R^i\Gamma(\mathcal{F}_3) & \xrightarrow{\delta^i_{\mathcal{F}}} & R^{i+1}\Gamma(\mathcal{F}_1) \\ R^i\Gamma\phi_3 \downarrow & \circlearrowleft & \downarrow R^i\Gamma\phi_1 \\ R^i\Gamma(\mathcal{G}_3) & \xrightarrow{\delta^i_{\mathcal{G}}} & R^{i+1}\Gamma(\mathcal{G}_1) \end{array}$$

(4) [Γ-acyclicity]

For every injective object $\mathcal{J}$ in $Sh(X)$ & each $i\geq 0$ we have $R^i\Gamma(\mathcal{J})=0$.

<u>Definition</u>: $\mathcal{J}$ is <u>injective</u> in $Sh(X)$ if $Hom(-,\mathcal{J})$ is exact.
(always left exact!)

Equivalently: $\forall$ injective morphism $A \xrightarrow{i} B$ & $\forall$ $A \xrightarrow{f} \mathcal{J}$ morphism $\exists$ $\tilde{f}: B \longrightarrow \mathcal{J}$ lifting f ($\tilde{f}|_A = f$)

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[dl, dashed, "\exists \tilde{f}"] \\ & \mathcal{J} & \end{tikzcd}$$

<u>Theorem</u> [Godement] $Sh(X)$ has enough injectives.
($\forall\, \mathcal{F}\in Sh(X)$ $\exists$ $\mathcal{J}\in Sh(X)$ injective with $0\to\mathcal{F}\to\mathcal{J}$ exact.)

<u>Q</u>: How to build $R^i\Gamma$?

<u>A</u>: Let's start with $i=1$. Say we have $\mathcal{F}_2$ with $R^1\Gamma(\mathcal{F}_2)=0$. Then, using (2) we get

$$0 \longrightarrow \mathcal{F}_1(X) \xrightarrow{\varphi_X} \mathcal{F}_2(X) \xrightarrow{\psi_X} \mathcal{F}_3(X) \xrightarrow{\delta^1} R^1\Gamma(\mathcal{F}_1) \longrightarrow R^1\Gamma(\mathcal{F}_2)=0 \longrightarrow \cdots$$

Exactness at $R^1\Gamma(\mathcal{F}_1)$ gives $R^1\Gamma(\mathcal{F}_1) = \operatorname{coker}(\psi_X)$.

<u>Issue</u>: What do we mean by $R^1\Gamma(\mathcal{F}_2)=0$?

<u>A</u>: If every ses $0\to \boxed{\mathcal{F}_2} \to \mathcal{G}_2 \to \mathcal{H}_2 \to 0$ starting with $\mathcal{F}_2$ the sequence $0\to\mathcal{F}_2(X)\to\mathcal{G}_2(X)\to\mathcal{H}_2\to 0$ remains exact, then $R^1\Gamma(\mathcal{F}_2)=0$.

<u>Note</u>: (4) ensure that $\mathcal{F}_2$ injective in $Sh(X)$ satisfies this, but it is not the only option ("Flabby" or "Flasque" sheaves will do)

↳ every section on an open U of X, extends to X.

<u>Q</u>: How to define $R^k\Gamma(\mathcal{F})$ for $\mathcal{F}$ in $Sh(X)$?

<u>A</u>: • Build an injective resolution of $\mathcal{F}$: $0\to\mathcal{F}\to\mathcal{J}^0\to\mathcal{J}^1\to \cdots$

• Apply Γ to $0\to\mathcal{J}^0\to\mathcal{J}^1\to\cdots$ to get a complex

Then, $R^k\Gamma(\mathcal{F}) := H^k(0\to\mathcal{J}^0(X)\to\mathcal{J}^1(X)\to\mathcal{J}^2(X)\to\cdots)$ in $\mathcal{C}$

Overall construction: ① we replace $Sh(X) \xrightarrow{\Gamma} \mathcal{C}$ by a functor between the categories of complexes

$$K^\bullet(Sh(X)) \xrightarrow{\Gamma} K^\bullet(\mathcal{C})$$

where
- Objects are complexes
- Morphisms are morphisms of complexes up to homotopy

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}_0 \arrow[r, "d_0^{\mathcal{F}}"] \arrow[d, shift right, "f_0"'] \arrow[d, shift left, "g_0"] & \mathcal{F}_1 \arrow[r, "d_1^{\mathcal{F}}"] \arrow[d, shift right, "f_1"'] \arrow[d, shift left, "g_1"] \arrow[dl, dotted, "\exists s_1"'] & \mathcal{F}_2 \arrow[r, "d_2^{\mathcal{F}}"] \arrow[d, shift right, "f_2"'] \arrow[d, shift left, "g_2"] \arrow[dl, dotted, "\exists s_2"'] & \cdots \arrow[dl, dotted, "\exists s_3"'] \\
0 \arrow[r] & \mathcal{G}_0 \arrow[r, "d_1^{\mathcal{G}}"'] & \mathcal{G}_1 \arrow[r, "d_2^{\mathcal{G}}"'] & \mathcal{G}_3 \arrow[r, "d_3^{\mathcal{G}}"'] & \cdots
\end{tikzcd}$$

f & g are morphisms between these 2 complexes.

We say f & g are **homotopic** if $\exists$ $s_i : \mathcal{F}_i \to \mathcal{G}_{i-1}$ for $i \geq 0$ with

$$f_i - g_i = d^{\mathcal{G}}_{i-1} \circ s_i + s_{i+1} d^{\mathcal{F}}_i \quad \text{for all } i \geq 0$$

② We have $\{H^n\}_{n \geq 0}$ functors in $K(\mathcal{C})$ that agree on homotopy-equivalent complexes in $K^\bullet(\mathcal{C})$

③ Injective resolutions yield an injection

$$Sh(X) \hookrightarrow K^\bullet(Sh(X))$$
$$\mathcal{F} \longmapsto (I^\bullet : 0 \to \mathcal{F} \to I_0 \to I_1 \to \dots)$$

Definition: $R^k\Gamma(\mathcal{F}) = H^k\left(0 \to \Gamma(X, I^0) \to \Gamma(X, I^1) \to \cdots\right)$

④ **Theorem:** Injective resolutions are unique up to homotopy.

$$\begin{tikzcd}
\mathcal{F} \arrow[d] & Sh(X) \arrow[d, hook] \arrow[rrr] & \mathcal{F} \arrow[r, mapsto] & & \{H^n(X,\mathcal{F})\}_{n \geq 0} \\
\mathcal{J}^\bullet & K^\bullet(Sh(X)) \arrow[r, "\Gamma"] & K^\bullet(\Gamma) \arrow[r, "H^n"] & \mathcal{C} & \| \\
& 0 \to I^0 \to I^1 \to \dots \arrow[r, mapsto] & \Gamma(X, \mathcal{J}^\bullet) \arrow[r, mapsto] & H^n(\Gamma(X, \mathcal{J}^\bullet)) =: R^n\Gamma(\mathcal{F}) &
\end{tikzcd}$$

Corollary: $H^k(X, \mathcal{F}) := R^k\Gamma(\mathcal{F})$ is independent of the resolution.

↳ sheaf cohomology

Disclaimer: This definition of $H^k(X, \mathcal{F})$ is not good for computing. Later, we will give a different construction (Čech cohomology) that is computationally friendly for Noetherian topological spaces (eg affine/projective varieties)