Lecture XVI: Projective Varieties I

Motivation: $\mathbb{A}^n_{\mathbb{K}}$ & affine varieties are not compact. Projectivization will compactify them, and will lead to better tools to study them (eg Intersection Theory, degree, etc.)

§1 Projective Space:

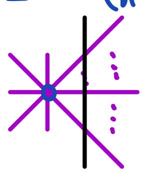
Fix $\mathbb{V}$ a finite-dimensional vector space over $\mathbb{K}$ ($\mathbb{V} \simeq \mathbb{K}^{n+1}$) ↳ choice of basis

Definition: The projectivization of $\mathbb{V}$ is

$$\mathbb{P}(\mathbb{V}) = \{\text{lines through } 0 \in \mathbb{V}\} \underset{\text{choice of basis}}{\simeq} \{(a_0, \ldots, a_n) \in \mathbb{K}^{n+1} \smallsetminus \{\underline{0}\} \mid \underline{a} \sim \lambda \underline{a} \ \forall \lambda \in \mathbb{K}^*\}$$

We usually write $[a_0 : \cdots : a_n]$ for the point in $\mathbb{P}(\mathbb{V})$ represented by $(a_0, \ldots, a_n)$

Examples: ① $\mathbb{P}^n_{\mathbb{K}} = \mathbb{P}(\mathbb{K}^{n+1})$

② $\mathbb{P}^1_{\mathbb{K}} =$  Each line meets $x=1$ except $x=0$. These intersections are all distinct!

We identify $\mathbb{P}^1$ with $\mathbb{A}^1 \sqcup \{\text{line } x=0\} = \mathbb{A}^1 \sqcup \{pt\}$. The pt is in the closure of $\mathbb{A}^1$. We call it the point at infinity.

More generally, we write:

$$\mathbb{P}^n_{\mathbb{K}} = \{[a_0 : \cdots : a_n] \in \mathbb{P}^n_{\mathbb{K}} \mid a_0 \neq 0\} \sqcup \{[a_0 : \cdots : a_n] \in \mathbb{P}^n_{\mathbb{K}} \mid a_0 = 0\}$$

$$= \mathbb{A}^n_{\mathbb{K}} \sqcup \boxed{\mathbb{P}^{n-1}_{\mathbb{K}}}$$

→ Name: Hyperplane at infinity for $\mathbb{A}^n_{\mathbb{K}}$.

Inductively, we get a decomposition:

$$\mathbb{P}^n_{\mathbb{K}} = \mathbb{A}^n_{\mathbb{K}} \sqcup \mathbb{A}^{n-1}_{\mathbb{K}} \sqcup \cdots \sqcup \mathbb{A}^1_{\mathbb{K}} \sqcup \{pt\}$$

Lemma 1: $\mathbb{P}^n_{\mathbb{K}}$ has a natural cover by affine spaces $U_0, \ldots, U_n$, each isomorphic to $\mathbb{A}^n$. More precisely:

$$U_j = \{[a_0 : \cdots : a_n] \mid a_j \neq 0\} = \{(\tfrac{a_0}{a_j}, \ldots, \tfrac{a_j}{a_j}, \ldots, \tfrac{a_n}{a_j})\} \simeq \mathbb{A}^n_{\mathbb{K}}$$

($\frac{a_j}{a_j} = 1$; omit j^{th} entry (= 1 always))

Alternatively: $U_j = \mathbb{P}^n \smallsetminus H_j$ where $H_j = \{[\underline{a}] \in \mathbb{P}^n \mid a_j = 0\}$

We call U_j's the standard affine patches

- Next, we put a topology on $\mathbb{P}^n_{\mathbb{K}}$ exploiting the covering description of $\mathbb{P}^n$ by distinguished affines.

<u>Definition</u>: $X \subseteq \mathbb{P}^n$ is <u>closed in the Zariski Topology</u> if, and only if, $X \cap U_i$ is closed in $U_i \simeq \mathbb{A}^n$ in the Zariski Topology $\forall i = 1, \dots, n$.

<u>Lemma 2</u>: The Zariski Topology on $\mathbb{P}^n_{\mathbb{K}}$ is the quotient topology for the subspace topology on $\mathbb{A}^{n+1}_{\mathbb{K}} \setminus \{0\}$ relative to the Zariski Topology of $\mathbb{A}^{n+1}_{\mathbb{K}}$.

<u>Proof</u>: In the Quotient Topology induced by $\mathbb{A}^{n+1}_{\mathbb{K}} \setminus \{0\} \xrightarrow{\pi} \mathbb{P}^n_{\mathbb{K}}$, $\mathcal{U} \subseteq \mathbb{P}^n_{\mathbb{K}}$ is open $\Leftrightarrow$ $\pi^{-1}(\mathcal{U}) \subseteq \mathbb{A}^{n+1}_{\mathbb{K}} \setminus \{0\}$ is open. The rest of the argument follows from this & $\pi^{-1}(\mathcal{U} \cap U_i) = \{\lambda \underline{u} : \underline{u} \in \mathcal{U} \ \& \ u_i = 1, \lambda \in \mathbb{K}^*\} = \pi^{-1}(\mathcal{U}) \setminus V(x_i)$ in $\mathbb{A}^{n+1}_{\mathbb{K}}$. □

<u>Remark</u>: We can define the Euclidean topology on $\mathbb{P}^n_{\mathbb{C}}$ using the same quotient + subspace topology construction, but relative to the Euclidean Topology on $\mathbb{C}^{n+1}$.
As with the affine case, the Euclidean Topology is finer that the Zariski one.

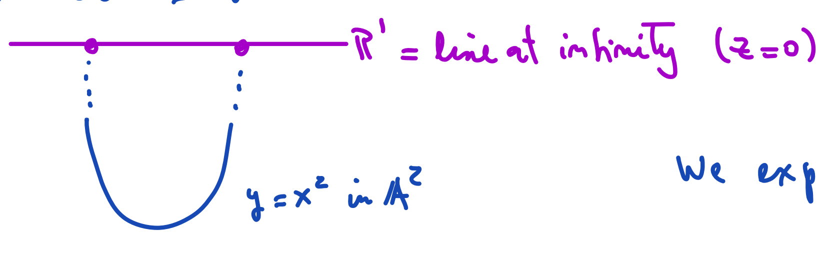
<u>Proposition 1</u>: $\mathbb{P}^n_{\mathbb{C}}$ is compact in the Euclidean Topology.

<u>Proof</u>: $S^n \subseteq \mathbb{C}^{n+1}$ is compact & $S^n \twoheadrightarrow \mathbb{P}^n_{\mathbb{C}}$, $\underline{a} \longmapsto [\underline{a}]$ is continuous in the Euclidean Top.

Q: How to compute (Zariski) closures in $\mathbb{P}^n$?

<u>Example</u>: Take $\mathbb{P}^2 = \mathbb{P}^2_{[x:y:z]}$ & $V^0 = \{y - x^2 = 0\} \subseteq U_2 \simeq \mathbb{A}^2 \subseteq \mathbb{P}^2$. What is $\overline{V^0}$?

Q: what is $\overline{V^0}$?



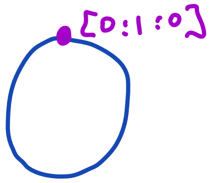
We expect $\overline{V^0} = V \cup \{2 \text{ pts}\}$

On U_2 we have $[x:y:z] = [\frac{x}{z} : \frac{y}{z} : 1]$ so $y = \frac{Y}{Z}$, $x = \frac{X}{Z}$.

$$y - x^2 = \frac{Y}{Z} - \left(\frac{X}{Z}\right)^2 = \frac{1}{Z^2}(ZY - X^2) = 0 \underset{\substack{Z \neq 0 \\ \text{in } U_2}}{\Longleftrightarrow} ZY - X^2 = 0$$

So extra points become $ZY - X^2 = Z = 0$, ie $Z = 0 = X^2$ so we get 1 double point, namely $[0:1:0]$. This point lies in $U_1 \setminus (U_0 \cup U_2)$.

$\Rightarrow \overline{V}^0$:



Claim: $V := \{ ZY - X^2 = 0 \} = \overline{V^0} \subseteq \mathbb{P}^2$ & $V = V^0 \cup \{[0:1:0]\}$

Pf/ To check V is closed, we need to compute $V \cap U_0$, $V \cap U_1$, $V \cap U_2$ & check they are closed in each $\mathbb{A}^2$.

- $V \cap U_0$: Use coordinates: $y = \frac{Y}{X}$, $z = \frac{Z}{X}$ so $ZY - X^2 = X^2(zy - 1) = 0$ ($X \neq 0$)

  so $V \cap U_0 = V(zy-1) \subseteq \mathbb{A}^2$ which is Zariski closed.

- $V \cap U_1$: Similarly $ZY - X^2 = Y^2\left(\frac{Z}{Y} - \left(\frac{X}{Y}\right)^2\right)$, so $V \cap U_1 = V(z - x^2) \subseteq \mathbb{A}^2$ is closed
- $V \cap U_2$: $ZY - X^2 = Z^2\left(\frac{Y}{Z} - \left(\frac{X}{Z}\right)^2\right)$ so $V \cap U_2 = V(y - x^2) \subseteq \mathbb{A}^2$ is closed

Since $V \cap U_2 = V^0$ closed in U_2, we get $\overline{V^0} \subseteq V$.

$V^0 \cap U_1 = \{ ZY - X^2 = 0,\ Y \neq 0,\ Z \neq 0 \}$

$\Rightarrow \overline{V^0 \cap U_1}^{U_1} = \{ ZY - X^2,\ Y \neq 0 \} = V \cap U_1$.

Similarly $\overline{V^0 \cap U_0}^{U_0} = \{ ZY - X^2,\ X \neq 0 \} = V \cap U_0$.

Conclude: $\overline{V^0} = V$. □

The example above shows us how we can define projective varieties.

§2 Projective Varieties:

Q: When does a polynomial in $\mathbb{K}[x_0, \dots, x_n]$ has a well-defined zero locus?

A: It should be invariant under non-zero scaling of $x_0 \dots x_n$!

Ex: $F = ZY - X^2$ is ok $F(\lambda x, \lambda y, \lambda z) = (\lambda z)(\lambda y) - (\lambda x)^2 = \lambda^2 (ZY - X^2) = \lambda^2 F(x,y,z)$

$F = X + 1$ is not ok. $F(-1:0) = 0$ but $F(-\lambda, 0) \neq 0$ if $\lambda \neq 1$.

Lemma 3: If $F = F_d$ is homogeneous of degree d ($\deg x_i = 1\ \forall i$), then

$$F(\lambda x_0, \dots, \lambda x_n) = \lambda^d F(x_0, \dots, x_n).$$

Corollary 1: $V_{proj}(F) \subseteq \mathbb{P}^n_{\mathbb{K}}$ is well-defined if, and only if, F is homogeneous!

$\{[\underline{a}] \in \mathbb{P}^n_{\mathbb{K}} : F(\underline{a}) = 0$ for any representative $\underline{a}$ of $[\underline{a}]\}$

Remark: F does not define a function $\mathbb{P}^n_{\mathbb{K}} \longrightarrow \mathbb{A}^1_{\mathbb{K}}$ ($F([\underline{a}])$ is not well-defined), but it's vanishing locus is.

This leads naturally to consider vanishing sets of homogeneous polynomials

Definition: A projective subvariety of $\mathbb{P}^n$ is the zero locus of a collection S of homogeneous polynomials in $\mathbb{K}[x_1, \ldots, x_n]$. (name: a homogeneous set)

$$\boxed{V_{proj}(S) = \{[a_0 : \ldots : a_n] \in \mathbb{P}^n : f(a_0, \ldots, a_n) = 0 \quad \forall f \in S\}}$$

↳ for projective vanishing locus.

Examples: ⓪ $\emptyset = V_{proj}(\{1\}) = V_{proj}(\{x_0, \ldots, x_n\})$ $(\underline{0} \notin \mathbb{P}^n_{\mathbb{K}})$

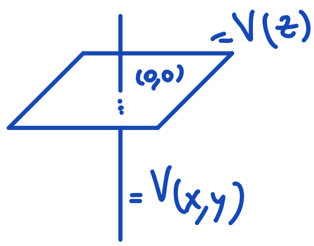
Definition: $I^0 = \langle x_0, \ldots, x_n \rangle$ is called the irrelevant ideal of $\mathbb{K}[x_0, \ldots, x_n]$

Remark: The name comes from the fact that $V_{proj}(I^0) = \emptyset$ in $\mathbb{P}^n$.

Examples: ① $\mathbb{P}^n_{\mathbb{K}} = V_{proj}(\{0\})$

② Line $\mathbb{P}^1$ ______ in $\mathbb{P}^2$ is $V_{proj}(\{x_2\}) = V_{proj}(\{x_2^2\})$

③ $V(xz, yz)$ in $\mathbb{A}^3$ is



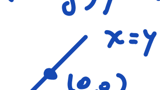
2 components

So $V_{proj}(xz, yz)$ in $\mathbb{P}^2 = \underset{(z=0)}{\mathbb{P}^1} \cup \underset{[0:0:1]}{\{pt\}}$

Note: "dimensions" dropped by one when projectivizing.

Remark: Ex 3 is reducible $V = V_1 \cup V_2$ with $V_i \subsetneq V$ for $i = 1, 2$.

Examples: ④ $V_{proj}(x(x-y), y(x-y))$ in $\mathbb{P}^1 = V_{proj}(x-y) = \{[1:1]\}$ a pt.

The vanishing set in $\mathbb{A}^2$ was  .

Definition: An ideal $I \subseteq \mathbb{K}[x_0,\dots,x_n]$ is homogeneous if it is generated by homogeneous polynomials.

Lemma 4: Fix an ideal $I \subseteq \mathbb{K}[x_0,\dots x_n]$. The following are equivalent:

(1) I is homogeneous

(2) If $f \in I$ & $f = f_0 + \dots + f_d$ is its decomposition into homogeneous pieces, [f_r is homogeneous of degree r] we have $f_r \in I$ $\forall$ $r = 0,\dots,d$.

Proof: (1) $\Rightarrow$ (2) Set $I = \langle g_1,\dots,g_m\rangle$ where g_i is homogeneous of degree d_i.

Write $f \in I$ as $f = \sum_{i=1}^{m} h^{(i)} g_i$. Decomposing each $h^{(i)}$ into homogeneous pieces we see $f_r = \sum_{\substack{1\leq j\leq m \\ d_j \leq r}} h^{(j)}_{r-d_j} \cdot g_j \in I$

(2) $\Rightarrow$ (1) Pick a set of generators for I say $\{h^{(1)},\dots,h^{(m)}\}$. By (2) each homogeneous piece of $h^{(i)}$ lies in I. Thus:

$$I = \langle h^{(0)},\dots,h^{(m)}\rangle = \langle h^{(0)}_0,\dots,h^{(0)}_{d_0}, h^{(1)}_0,\dots,h^{(1)}_{d_1},\dots,h^{(m)}_0,\dots,h^{(m)}_{d_m}\rangle$$

where d_i = total degree of $h^{(i)}$.

We conclude that I is generated by homogeneous polynomials, so it is homogeneous. □

. Just as in the case of affine varieties, we have several useful properties. The proofs are verbatim, once we restrict to homogeneous sets.

Lemma 5: If $S_1 \subseteq S_2 \subseteq \mathbb{K}[x_1,\dots,x_n]$ are homogeneous, we have $V_{proj}(S_1) \supseteq V_{proj}(S_2)$

Lemma 6: If $S_1, S_2 \subseteq \mathbb{K}[x_1,\dots,x_n]$ are homogeneous we have

$$V_{proj}(S_1) \cup V_{proj}(S_2) = V_{proj}(S_1 S_2)$$

where $S_1 \cdot S_2 = \{fg : f \in S_1 \text{ \& } g \in S_2\}$

Proof: ($\subseteq$) $\underline{a} \in V(S_1) \Rightarrow f(\underline{a}) = 0$ $\forall f \in S_1$

$\Rightarrow f \cdot g = 0$ $\forall f \in S_1$ & $\forall g \in S_2$, hence $\underline{a} \in V_{proj}(S_1 S_2)$

• Same ideas give $V_{proj}(S_2) \subseteq V_{proj}(S_1 S_2)$

($\supseteq$) We prove the contrapositive. Say $\underline{a} \notin (V_{proj}(S_1) \cup V_{proj}(S_2))$. So $\exists f \in S_1, g \in S_2$ with $f(\underline{a}) \neq 0$ & $g(\underline{a}) \neq 0$. But then $(fg)(\underline{a}) \neq 0$, meaning $\underline{a} \notin V_{proj}(S_1 S_2)$ □

Lemma 7: If Λ is any index set & $S_i \subseteq \mathbb{K}[x_1, \dots x_n]$ are homogeneous for for any $i \in \Lambda$, then
$$\bigcap_{i \in \Lambda} V_{proj}(S_i) = V_{proj}\left(\bigcup_{i \in I} S_i\right)$$

Proof: $\underline{a} \in \bigcap_{i \in \Lambda} V_{proj}(S_i) \iff \forall i \, \& \, \forall f \in S_i$ we have $f(\underline{a}) = 0$.
$\iff \forall f \in \bigcup_{i \in \Lambda} S_i$ we have $f(\underline{a}) = 0 \iff \underline{a} \in V_{proj}\left(\bigcup_{i \in \Lambda} S_i\right)$

Remark: $\emptyset = V_{proj}(I^0)$, $\mathbb{P}^n = V_{proj}(0)$ combined with Lemmas 6 & 7 say that projective subvarieties define closed sets for a topology on $\mathbb{P}^n$. This is precisely the Zariski topology we defined earlier. We will discuss this next time.

Proposition 2: $V_{proj}(S) = V_{proj}(\langle S \rangle)$, where $\langle S \rangle$ = ideal in $\mathbb{K}[\underline{x}]$ generated by S.

Proof: We prove the double inclusion.

($\supseteq$) $S \subseteq \langle S \rangle$ so by Lemma 4 we have $V_{proj}(S) \supseteq V_{proj}(\langle S \rangle)$

($\subseteq$) Say $f \in \langle S \rangle$ so $\exists k \geq 1, \exists f_1, \dots, f_k \in S$ & $g_1, \dots, g_k \in \mathbb{K}[\underline{x}]$ with $f = \sum_{i=1}^{k} g_i f_i$

If $\underline{a} \in V_{proj}(S)$, then $f_i(\underline{a}) = 0 \ \forall i = 1, \dots, k$. In particular,
$$f(\underline{a}) = \left(\sum_{i=1}^{k} g_i f_i\right)(\underline{a}) = \sum_{i=1}^{k} g_i(\underline{a}) f_i(\underline{a}) = \sum_{i=1}^{k} g_i(\underline{a}) \cdot 0 = 0$$

Conclusion: $f(\underline{a}) = 0 \ \forall f \in \langle S \rangle$ ie $\underline{a} \in V_{proj}(\langle S \rangle)$ □

Corollary 2: Any projective subvariety of $\mathbb{P}^n$ is defined by a finite list of homogeneous polynomials.

Proof: Homogeneous Ideals are finitely generated by the Noetherianness of $\mathbb{K}[x_0, \dots x_n]$