Lecture XVII: Projective Varieties II

Recall V $(n+1)$-dimensional vector space $V \simeq \mathbb{K}^{n+1}$, then

$$\mathbb{P}(V) = \{\text{lines in } V\} \cong \mathbb{A}^{n+1}_{\mathbb{K}} \setminus \{0\} \big/ \underline{a} \sim \lambda \underline{a} \ \forall \lambda \in \mathbb{K}^* = \mathbb{P}^n_{\mathbb{K}}$$

- Decomposition in Affine spaces: $\mathbb{P}^n_{\mathbb{K}} = \mathbb{A}^n_{\mathbb{K}} \sqcup \mathbb{A}^{n-1}_{\mathbb{K}} \sqcup \cdots \sqcup \mathbb{A}^1_{\mathbb{K}} \sqcup \{pt\}$
- Covering by standard affine patches $U_j = \{\underline{a} \in \mathbb{P}^n_{\mathbb{K}} : a_j \neq 0\} \simeq \mathbb{A}^n_{\mathbb{K}}$.

Definition: A projective subvariety of $\mathbb{P}^n$ is the zero locus of a collection S of homogeneous polynomials in $\mathbb{K}[x_1, \ldots, x_n]$. (name: a homogeneous set)

$$\boxed{V_{proj}(S) = \{[a_0 : \ldots : a_n] \in \mathbb{P}^n : f(a_0, \ldots, a_n) = 0 \quad \forall f \in S\}}$$

Lemma: $V_{proj}(S) = V_{proj}(\langle S \rangle)$ & $\langle S \rangle$ is a homogeneous ideal.

Definition: The ideal $I^0 = \langle x_0, \ldots, x_n \rangle$ of $\mathbb{K}[x_0, \ldots x_n]$ is called the irrelevant ideal

By construction, $V_{proj}(I^0) = \emptyset \subseteq \mathbb{P}^n$.

§1. Ideals from projective varieties:

Definition: Given a subset $W \subseteq \mathbb{P}^n$, we define:

$$\boxed{I^h(W) = \{f \in \mathbb{K}[x_0, \ldots, x_n] : f(\underline{a}) = 0 \quad \forall \underline{a} \in W\} \subseteq \mathbb{K}[x_0, \ldots, x_n]}$$

(This makes sense even if W is not a projective variety)

Proposition 1: $I^h(W)$ is a homogeneous ideal of $\mathbb{K}[\underline{x}]$ if $\mathbb{K}$ is infinite.

Proof: Need to show 3 properties of ideals (these are valid even if $\mathbb{K}$ is finite)

(1) $0 \overset{?}{\in} I(W)$: $0(\underline{a}) = 0 \quad \forall \underline{a} \in V$ ✓

(2) $f, g \in I(W) \overset{?}{\Longrightarrow} f+g \in I(V)$

$f(\underline{a}) = 0$ & $g(\underline{a}) = 0 \quad \forall \underline{a} \in W$ by def, so $(f+g)_{(\underline{a})} = f(\underline{a}) + g(\underline{a}) = 0 + 0 = 0 \quad \forall \underline{a} \in W$ ✓

(3) $f \in I^h(W)$, $h \in \mathbb{K}[\underline{x}] \overset{?}{\Longrightarrow} h \cdot f \in I(W)$

$f(\underline{a}) = 0 \quad \forall \underline{a} \in W$, so $(hf)_{(\underline{a})} = h(\underline{a}) f(\underline{a}) = h(\underline{a}) \cdot 0 = 0 \quad \forall \underline{a} \in W$ ✓ □

- Next, we show $I^h(W)$ is a homogeneous ideal if $\mathbb{K}$ is infinite.

Pick $f \in W$, so $f(a_0, \ldots a_n) = f(\lambda a_0, \ldots, \lambda a_n) = 0 \quad \forall (a_0, \ldots a_n) \in W$

Writing $f = f_0 + \cdots + f_d$ for $d = \deg(f)$, we get

$$0 = f(\underline{a}) + f_1(\underline{a}) + \cdots + f_d(\underline{a}) = f(\underline{a}) + \lambda_1 f_1(\underline{a}) + \lambda^2 f_2(\underline{a}) + \cdots + \lambda^d f_d(\underline{a})$$

Fix $\underline{a} \in W$. If $f_r(\underline{a}) \neq 0$ for some r then

$$G_{\underline{a}}(\lambda) = f(\underline{a}) + \lambda_1 f_1(\underline{a}) + \cdots + \lambda^d f_d(\underline{a}) \in \mathbb{K}[\lambda] \setminus \{0\}$$

is constantly 0 on $\mathbb{K}^*$, which is dense in $\mathbb{K}$. Since polynomials are continuous functions $\mathbb{A}^1_{\mathbb{K}} \longrightarrow \mathbb{A}^1_{\mathbb{K}}$, we conclude $G_{\underline{a}}(\lambda) = 0 \quad \forall \lambda \in \mathbb{K}$.

Since $\mathbb{K}$ is infinite, this forces $G_{\underline{a}} \equiv 0$, which contradicts $f_r(\underline{a}) \neq 0$

Conclude: $f_r(\underline{a}) = 0 \quad \forall \underline{a} \in W \quad \forall r = 0, \ldots, d$, so $f_r \in I^h(W) \ \forall r$.

By Lemma 4 §16.2, $I^h(W)$ is homogeneous. □

- We have analogs of Lemmas 1, 2 & 3 of § 3.1. Again, with verbatim proofs.

Lemma 1: If $W_1 \subseteq W_2$ in $\mathbb{P}^n$, then $I^h(W_1) \supseteq I^h(W_2)$.

Lemma 2: If W_1, W_2 are subsets of $\mathbb{P}^n$ we have:

$$I^h(W_1 \cup W_2) = I^h(W_1) \cap I^h(W_2)$$

Proof: $f \in I^h(W_1 \cup W_2) \iff \forall \underline{a} \in W_1 \cup W_2$ we have $f(\underline{a}) = 0$

$\iff \forall \underline{a} \in W_1$ we have $f(\underline{a}) = 0$ & $\forall a \in W_2$ we have $f(\underline{a}) = 0$

$\iff f \in I^h(W_1)$ & $f \in I^h(W_2) \iff f \in I^h(W_1 \cap W_2)$ □

Lemma 3: If W_1, W_2 are subsets of $\mathbb{P}^n$ we have:

$$I^h(W_1 \cap W_2) \supseteq I^h(W_1) + I^h(W_2)$$

Proof: Since $I^h(W_1 \cap W_2)$ is an ideal, it suffices to check that $I^h(W_i) \subseteq I^h(W_1 \cap W_2)$ for $i = 1, 2$, but this follows from Lemma 8 §16.2 since $W_i \supseteq W_1 \cap W_2$ □

⚠ The inclusion $\supseteq$ can be strict even if W_1 & W_2 are projective varieties.

§2. Duality between homogeneous ideals & projective varieties:

The results from §1 & §16.2 yield the following Basic Duality for projective subvarieties of $\mathbb{P}^n$:

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$$\left\{\begin{array}{c}\text{Subvarieties}\\ \text{of } \mathbb{P}^n\end{array}\right\} \underset{V_{proj}}{\overset{I^h}{\rightleftarrows}} \left\{\begin{array}{c}\text{homogeneous}\\ \text{ideals } \mathfrak{a}\\ \text{of } \mathbb{K}[x_0,\dots,x_n]\end{array}\right\}$$

Next, we discuss how I^h & V_{proj} interact with each other.

Proposition 2: If $W \subseteq \mathbb{P}^n$ is a subvariety, then $V_{proj}(I^h(W)) = W$.

Proof: ($\supseteq$) is easy to check: If $\underline{a} \in W$, then $f(\underline{a}) = 0$ $\forall f \in I^h(W)$ by definition of $I^h(W)$, meaning $\underline{a} \in V_{proj}(I^h(W))$.

($\subseteq$) is also easy to check. Since W is projective subvariety of $\mathbb{P}^n$, then $W = V_{proj}(S)$ for a finite set $S = \{f_1, \dots, f_k\}$ of homogeneous polynomials in $\mathbb{K}[x_0, \dots, x_n]$ by Corollary 2 §16.2. Thus, we have $S \subseteq I^h(W)$ by definition of S.

Lemma 5 yields $W = V_{proj}(S) \supseteq V_{proj}(I^h(W))$ □

Corollary 1: For 2 varieties W_1 & W_2 we have $W_1 \subseteq W_2 \iff I^h(W_1) \supseteq I^h(W_2)$

Proof: Combine Proposition 2 & Lemma 8 §16.2

Proposition 3: For any ideal $\mathfrak{a}$ of $\mathbb{K}[x_1, \dots, x_n]$ we have $I^h(V_{proj}(\mathfrak{a})) \supseteq \mathfrak{a}$

Proof: Pick any $\underline{a} \in W = V_{proj}(\mathfrak{a})$ & $f \in \mathfrak{a}$. Then, $f(\underline{a}) = 0$ & so $f \in I^h(V_{proj}(\mathfrak{a}))$ □

In particular, $\mathfrak{a} \subseteq I^h(V_{proj}(\mathfrak{a}))$.

⚠ This is not a 1-to-1 correspondence even if $\mathbb{K}=\overline{\mathbb{K}}$. (ideals are restricted!)

Ex: $\emptyset = V(\{x,y\}) = V(1)$ in $\mathbb{P}^1$

Lemma 4: $I(W)$ is a radical ideal for any $W \subseteq \mathbb{A}^n$.

Proof: If $f^N \in I^h(W)$ then $(f^N)(\underline{a}) = (f(\underline{a}))^N = 0 \quad \forall \underline{a} \in W$. But $\mathbb{K}$ is a field, so this forces $f(\underline{a})=0 \quad \forall \underline{a} \in W$, ie $f \in I^h(W)$ □

Corollary 2: For any homogeneous ideal $\mathfrak{a} \subseteq \mathbb{K}[x_0,\ldots,x_n]$, we have

$$I^h(V_{proj}(\mathfrak{a})) \supseteq \sqrt{\mathfrak{a}}.$$

Remark: $\sqrt{\mathfrak{a}}$ is a homogeneous ideal whenever $\mathfrak{a}$ is homogeneous.

The example shows that even when $\overline{\mathbb{K}}=\mathbb{K}$, radicals are not the only subtle point!

§3. Irreducible decomposition of projective varieties:

Definition: A variety $V \subseteq \mathbb{P}^n$ is irreducible if for every expression of V as a union $V = V_1 \cup V_2$ we have either $V_1 = V$ or $V_2 = V$.

A projective subvariety is reducible if it is not irreducible.

Example: ① $\mathbb{P}^n$ is irreducible (same proof as the one that shows $\mathbb{A}^n$ is irreducible)

② $V_{proj}(xz, yz) = \underbrace{V_{proj}(x,y)}_{=\{[0:0:1]\}} \cup \underbrace{V_{proj}(z)}_{=\mathbb{P}^1}$ in $\mathbb{P}^2$ is reducible.

Theorem 1: Every projective variety is a finite union of irreducible projective varieties.

Proof Given $W \subseteq \mathbb{P}^n$ projective variety, we use the same bisection argument we invoked for affine varieties to build an infinite descending chain

$$W \supsetneq W_1 \supsetneq W_2 \supsetneq \cdots$$

of reducible projective varieties in $\mathbb{P}^n$.

Applying I^h we obtain an infinite strictly increasing chain of ideals

$$I^h(W) \subsetneq I^h(W_1) \subsetneq I^h(W_2) \subsetneq \cdots$$

in the Noetherian ring $\mathbb{K}[x_0,\ldots,x_n]$, which cannot happen. □

Challenges: ① How to detect irreducibility?

② How to perform irreducible decompositions in practice?

For ① we have an easy characterization.

Proposition 4: A variety $W \subseteq \mathbb{P}^n$ is irreducible $\iff$ $I^h(W)$ is a prime ideal

Proof We prove both implications

($\Rightarrow$) Pick $f, g \in \mathbb{K}[\underline{x}]$ with $fg \in I^h(W)$.

Since $I^h(W)$ is homogeneous, by induction on $\max\{\deg f, \deg g\}$, we can reduce to the case when both f & g are homogeneous.

Then $W = V_{proj}(\langle I^h(W), f\rangle) \cup V_{proj}(\langle I^h(W), g\rangle)$ will give a decomposition

If $f, g \notin I^h(W)$, this decomposition would be nontrivial, contradicting our irreducibility assumption on W.

($\Leftarrow$) We argue by contradiction & assume $W = W_1 \cup W_2$ is a nontrivial decomposition of W. In particular, we know that $W_1 \not\subseteq W_2 \not\subseteq W_1$. Equivalently, by Corollary 1, this gives $\underset{\mathfrak{a}_1}{\underset{\shortparallel}{I^h(W_1)}} \not\subseteq \underset{\mathfrak{a}_2}{\underset{\shortparallel}{I^h(W_2)}} \not\subseteq I^h(W_1)$. So $\exists f \in \mathfrak{a}_1 \smallsetminus \mathfrak{a}_2$ & $g \in \mathfrak{a}_2 \smallsetminus \mathfrak{a}_1$.

By Lemma 4, we know $I^h(W_1 \cup W_2) = \mathfrak{a}_1 \cap \mathfrak{a}_2 \supseteq \mathfrak{a}_1 \cdot \mathfrak{a}_2$

true for any pair of ideals

Then $fg \in \mathfrak{a}_1 \cdot \mathfrak{a}_2 \subseteq I^h(W)$ & $f, g \notin I^h(W)$ contradiction our assumption that $I^h(W)$ was prime. $\square$

For ② we will translate a decomposition of W into a decomposition of the (radical) ideal $I(W)$ as an intersection of homog prime ideals of $\mathbb{K}[x_1 \ldots x_n]$

We do this via primary decompositions of proper ideals on a $\mathbb{N}_0$-graded ring R

The only thing to show is that primary components of homogeneous ideals are always homogeneous & the characterization of associated primes involves homogeneous elements. More precisely:

STEP 1: Decompose proper homogeneous ideals into finite intersections of homogeneous irreducible ideals (Technique: Zorn's Lemma)

STEP 2: Show that irreducible $\Rightarrow$ primary for homogeneous ideals

STEP 3: Characterize associated primes to a homogeneous ideal $\mathfrak{a}$ of R as $\{\sqrt{(\mathfrak{a}:x)} : x \text{ homogeneous } \& \sqrt{(\mathfrak{a}:x)} \text{ is prime}\}$.

- See HW 5 for details.

§4. Projective Nullstellensatz:

Q: Given $I \subseteq \mathbb{K}[x_0,\dots,x_n]$ homogeneous ideal, what is $V(I) \subseteq \mathbb{A}^{n+1}_{\mathbb{K}}$?

A: It is a cone!

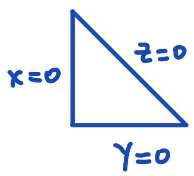
Definition: A closed subset $C \subseteq \mathbb{A}^{n+1}$ is a cone if the following condition holds

$$\underline{a} \in C \iff \lambda \underline{a} = (\lambda a_0, \dots, \lambda a_n) \in C \;\; \forall \lambda \in \mathbb{K}$$

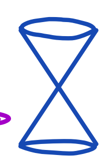
Remark: $\{0\} \in C$ whenever $C \neq \emptyset$ since $0 \cdot \underline{a} = 0$.

Definition: Given a projective subvariety $W \subseteq \mathbb{P}^n$ we define the affine cone $C(W) \subseteq \mathbb{A}^{n+1}$ as the set $\{(a_0,\dots,a_n) \mid [\underline{a}] \in W\} \cup \{0\} = \pi^{-1}(W) \cup \{0\}$, where $\pi : \mathbb{A}^{n+1}_{\mathbb{K}} \setminus \{0\} \twoheadrightarrow \mathbb{P}^n$ is the natural projection

Example: (1) $W = \{[1:0]\} \subseteq \mathbb{P}^1$ $\quad C(W) = \{(\lambda, 0) \;\; \lambda \in \mathbb{K}\} =$ 

(2) 

$= \mathbb{P}^2 \supseteq W = V_{proj}\{x^2+y^2-z^2\}$ $\quad C(W) = V(x^2+y^2-z^2) \subseteq \mathbb{A}^3$

real picture → 

$W \cap U_z = \{x^2+y^2=1\}$ $\qquad W \cap V_{proj}(z) = \{[x:y:0] : x^2+y^2=0\} = \{[1:i:0], [i:1:0]\}$

$C(W) \cap \{z=0\} = \{2 \text{ lines}\} = C(\{[1:i:0], [i:1:0]\})$.

Lemma 5: If $W = V_{proj}(I)$ for I homogeneous, then $C(W) = V(I) \subseteq \mathbb{A}^{n+1}$.

Proof: The proof is straightforward ($I \subseteq I(C(W))$ by definition so $V(I) \supseteq C(W)$; $V(I) \subseteq C(W)$ is clear).

Proposition 5: Let $X \subseteq \mathbb{A}^{n+1}_{\mathbb{K}}$ be a non-empty Zariski closed set. Assume X is cone.

If $\mathbb{K}$ is infinite, then:

(1) $X = V(I)$ for a homogeneous ideal

(2) $\pi(X \setminus \{0\}) = V_{proj}(I)$.

Proof: Set $I = I(X) \subseteq \mathbb{K}[x_0, \ldots, x_n]$

- To prove (1) it is enough to show that I is homogeneous whenever X is

Pick $f \in I$, so $f(\underline{a}) = 0$ $\forall a \in X$. Let $d = \deg(f)$. Since X is a cone.

we have $0 = f(\lambda a) = \sum_{k=0}^{d} \lambda^k f_k(\underline{a})$ $\forall \lambda$ where $f = f_0 + \cdots + f_d$ where f_k is

homogeneous of degree k. Set $G_{\underline{a}}(\lambda) = \sum_{k=0}^{d} f_k(a) \lambda^k \in \mathbb{K}[\lambda]$.

Since $G_{\underline{a}}(\lambda)$ vanishes on $\mathbb{K}$ (infinite), we have $G_{\underline{a}}(\lambda) = 0$ ie $f_k(\underline{a}) = 0$ $\forall k$.

Thus, for all $a \in X$ we have "$f(\underline{a}) = 0 \iff f_k(\underline{a}) = 0 \quad \forall k = 0, \ldots, \deg f$"

By Lemma 4 §16.2 we conclude that I is a homogeneous ideal.

- Statement (2) follows from (1) & the definitions of V_{proj} & π.

Theorem 2 (Projective Nullstellensatz): Assume $\overline{\mathbb{K}} = \mathbb{K}$ & let $\mathfrak{a}$ be a homogeneous ideal of $\mathbb{K}[x_0, \ldots, x_n]$. Then:

(1) $V_{proj}(\mathfrak{a}) = \emptyset \iff \mathfrak{a} \supseteq (x_0, \ldots, x_n)^m$ for some m

$\iff \sqrt{\mathfrak{a}} \supseteq I^0$

(2) If $V_{proj}(\mathfrak{a}) \neq \emptyset$, then $I^h V_{proj}(\mathfrak{a}) = \sqrt{\mathfrak{a}}$.