Lecture XVIII: Projective Varieties III

Recall: If $W \subseteq \mathbb{P}^n$ is given as $W = V_{proj}(I)$ for I homogeneous ideal of $\mathbb{K}[x_0,\ldots,x_n]$, then $C(W) = \pi^{-1}(W) \cup \{0\} \supseteq V(I) \subseteq \mathbb{A}^{n+1}_{\mathbb{K}}$ is the affine cone over W.
Furthermore, $C(W) = V(I)$ if, and only if, $W \neq \emptyset$.

§1 Projective Nullstellensatz:

Theorem 1 (Projective Nullstellensatz): Assume $\overline{\mathbb{K}} = \mathbb{K}$ & let $\mathfrak{a}$ be a homogeneous ideal of $\mathbb{K}[x_0,\ldots,x_n]$. Then:

(1) $V_{proj}(\mathfrak{a}) = \emptyset \iff \mathfrak{a} \supseteq (x_0, \ldots, x_n)^m$ for some $m \iff \sqrt{\mathfrak{a}} \supseteq I^0$

(2) If $V_{proj}(\mathfrak{a}) \neq \emptyset$, then $I^h V_{proj}(\mathfrak{a}) = \sqrt{\mathfrak{a}}$.

Proof: We prove the statement by reducing it to $C(W) \subseteq \mathbb{A}^{n+1}_{\mathbb{K}}$ where $W = V_{proj}(\mathfrak{a})$.
$\mathfrak{a} \neq (1)$ is a homogeneous ideal. By Lemma 5, $C(W) = V(\mathfrak{a}) \subseteq \mathbb{A}^{n+1}_{\mathbb{K}}$

By construction, $W = V_{proj}(\mathfrak{a}) = \emptyset \iff C(W) = \{0\}$

(1) ($\Rightarrow$) Since $C(W)$ is Zariski closed, the Strong Nullstellensatz implies:
$$I^0 = \mathfrak{m}_{\underline{0}} = I(\{0\}) = I(C(W)) = I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$$

($\Leftarrow$) If $\sqrt{\mathfrak{a}} \supseteq I^0$ then $W = V_{proj}(\mathfrak{a}) = V_{proj}(\sqrt{\mathfrak{a}}) \subseteq V_{proj}(I^0) = \emptyset$
Lemma 5 §16.2

(2) We prove the double inclusion. $I^h V_{proj}(\mathfrak{a}) \supseteq \sqrt{\mathfrak{a}}$ by Corollary 2.

For ($\subseteq$) we argue as follows. The strong Nullstellensatz implies $I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.
Since $V(\mathfrak{a}) \neq \{0\}$ we know that $I(V(\mathfrak{a})) \neq I^0$.
Now given $f \in I^h(W)$, we write $f = f_0 + \cdots + f_d$ where $d = \deg f$. Since $I^h(W)$ is homogeneous, we know $f_k \in I^h(W)$ $\forall k$. Thus, we may assume f is homogeneous.
Since $W \neq \emptyset$ we have $0 = f([\underline{a}])$ $\forall [\underline{a}] \in W$
Using the fact that f is homogeneous of degree d we conclude that
$$f(\lambda \underline{a}) = \lambda^d f(\underline{a}) = 0 \;\; \forall \lambda \in \mathbb{K}^* \quad \& \text{ by continuity } f(\underline{0}) = 0.$$

Since $f([\underline{a}]) = 0$ $\forall [\underline{a}] \in W$, we conclude that $f = 0$ on $C(W)$. Therefore,
$$f \in I(C(W)) = I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}} \qquad \square$$

Corollary 1: If $\overline{\mathbb{K}} = \mathbb{K}$ there is a 1-to-1 inclusion-reversing correspondence:

$$\boxed{\left\{\begin{array}{c}\text{projective subvarieties}\\ W \text{ of } \mathbb{P}^n_{\mathbb{K}},\ W \neq \emptyset\end{array}\right\} \underset{V_{proj}}{\overset{I^h}{\rightleftarrows}} \left\{\begin{array}{c}\text{homogeneous radical ideals}\\ I \text{ of } \mathbb{K}[x_0,\dots,x_n], I \neq I^0\end{array}\right\}}$$

§2. Homogeneous coordinate ring:

Assume $\mathbb{K}$ is an infinite field.

Definition: Given a non-empty projective subvariety $W \subseteq \mathbb{P}^n_{\mathbb{K}}$, we define its **homogeneous coordinate ring** as $S(W) = \mathbb{K}[x_0,\dots,x_n] / I^h(W)$

Example: $W = \mathbb{P}^n_{\mathbb{K}}$ then $S(W) = \mathbb{K}[x_0,\dots,x_n]$ $\quad I^h(W) = \langle 0 \rangle$.

Remark: Since $I^h(W)$ is homogeneous, we have $I^h(W) = \bigoplus_d (I^h(W))_d$ where $J_d := J \cap \underbrace{\mathbb{K}[x_0,\dots,x_n]_d}_{=: \{0\} \cup \{\text{homog. polynomials of degree } d\}}$ for any ideal J. We have a graded ideal.

Thus $S(W)$ becomes a graded ring with grading in $\mathbb{N}_0$. Indeed:

$$S(W) := \bigoplus_{d \geq 0} \frac{\mathbb{K}[x_0,\dots,x_n]_d}{(I^h(W))_d}$$

Note: $S_k \cdot S_\ell \subseteq S_{k+\ell}$ since $J_k J_\ell \subseteq J_{k+\ell}$ for $J = I^h(W)$ $\forall k, \ell$

$\mathbb{K}[\underline{x}]_k \mathbb{K}[\underline{x}]_\ell \subseteq \mathbb{K}[\underline{x}]_{k+\ell}$

This is the condition for the ring to be graded.

⚠ There are some major differences between $S(W)$ & $\mathbb{K}[C(W)]$

① Although they are the same ring, but in $\mathbb{K}[C(W)]$ we don't consider its grading.

② Element of $S(W)$ do not define functions on $W \subseteq \mathbb{P}^n$

The main issue is $g([a])$ is not well-defined! Even if g were homogeneous, say, of degree d, then $g(\lambda a) = \lambda^d g(a)$.

So $\bar{g} \in S(W)$ cannot define a function on W.

③ The only thing that we can do is to work with rational functions of degree 0, ie $\varphi = \frac{\overline{g}}{\overline{h}}$ where $g, h \in \mathbb{K}[x_0, \ldots, x_n]$ homogeneous of the same degree & $\overline{h} \neq 0$. (Rational functions are restricted!)

④ $S(W)$ is not intrinsic to W. This graded ring carries information about the embedding $X \hookrightarrow \mathbb{P}^n$.

Example: $S(\mathbb{P}^1) = \mathbb{K}[x_0, x_1]$ & $W = (\mathbb{P}^1 \hookrightarrow \mathbb{P}^2)$ via $[x_0 : x_1] \longmapsto [x_0^2 : x_0 x_1 : x_1^2]$ (Segre embedding)

$I^h(W) = \langle xz - y^2 \rangle$ & $S(W) = \frac{\mathbb{K}[x,y,z]}{\langle xz - y^2 \rangle}$.

$W \simeq \mathbb{P}^1$ but $S(\mathbb{P}^1) \not\simeq S(W)$ as (graded) rings ((RHS) is not a UFD, but the (LHS) is, so they cannot be isomorphic as rings)

§3. Homogenization & affinization:

We can view $\mathbb{A}^n_{\mathbb{K}} \xhookrightarrow{\varphi_i} \mathbb{P}^n_{\mathbb{K}}$ via $\varphi(\underline{a}) = [a_1 : a_2 : \cdots : a_i : 1 : a_{i+1} : \cdots : a_n]$

Definition: (1) Given $f \in \mathbb{K}[x_1, \ldots, x_n] \setminus \{0\}$ of degree d, we define the homogenization of f. to be the polynomial $f^h \in \mathbb{K}[x_0, \ldots, x_n]$ with $f^h(x_0, \ldots, x_n) = x_0^d f\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right)$

(2) Given an ideal $I \subseteq \mathbb{K}[x_1, \ldots, x_n]$, we define $I^h \subseteq \mathbb{K}[x_0, x_1, \ldots, x_n]$ to be the (homogeneous) ideal generated by $\{f^h : f \in I\}$

Example: $f = 1 + x + y^2 \rightsquigarrow f^h(z, x, y) = z^2\left(1 + \frac{x}{z} + \left(\frac{y}{z}\right)^2\right) = z^2 + zx + y^2$

⚠ If $I = \langle f_1, \ldots, f_r \rangle \not\Rightarrow I^h = \langle f_1^h, \ldots, f_r^h \rangle$

$I = \langle 1+x, y+x^2 \rangle$ $\quad (1+x)^h = z + x \quad ; \quad (y+x^2)^h = zy + x^2$

$(-x(1+x) + (y+x^2))^h = (y-x)^h = y - x \in I$ but $y - x \notin \langle z+x, zy + x^2 \rangle$

Definition: Given a homogeneous polynomial $F \in \mathbb{K}[x_0, \ldots, x_n]$, we define its affinization with respect to the variable x_k as $F^{(k)} = F(x_0, \ldots, x_{k-1}, 1, x_{k+1}, \ldots, x_n)$

Examples: (1) $F(x, y, z) = z^2 + zx + y^2 \rightsquigarrow F^{(3)} = 1 + x + y^2$

(2) $F(x,y,z) = x^2z + z^2y \rightsquigarrow f^{(3)} = x^2+y$ & $(f^{(3)})^h = x^2+zy \neq F$

Natural Questions:

① Given $V = V(f_1, \dots, f_m) \subseteq \mathbb{K}[x_0, \dots, \hat{x}_k, \dots x_n]$

Q: What is $V \subseteq \mathbb{A}^n \simeq U_k \subseteq \mathbb{P}^n$?

A: $V = V_{proj}(f_1^h, \dots, f_m^h) \cap U_k$

⚠ It is not necessarily true that $\overline{V}$ in $\mathbb{P}^n$ equals $V_{proj}(f_1^h, \dots f_m^h)$

↳ Zariski topology

Q2: How do we compute $\overline{V}$ in $\mathbb{P}^n$?

② Given $W = V_{proj}(F_1, \dots, F_m)$ where $F_1, \dots, F_m$ are homogeneous.

Q: What equations cut out $W \cap U_k$?

A: We affinize the equations! Thus, $W \cap U_k = V(F_1^{(k)}, \dots, F_m^{(k)})$

$F_i^{(k)} \in K[\frac{x_0}{x_k}, \dots, \overset{k}{\hat{1}}, \dots, \frac{x_n}{x_k}]$