Lecture XIX: Projective Varieties IV

Recall: $\mathbb{A}^n \longrightarrow U_k = \{[a] \in \mathbb{P}^n \mid a_k \neq 0\}$ for $k = 0, \ldots, n$

$$[a_0, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n] \longmapsto [a_0 : \cdots : a_{k-1} : 1 : a_{k+1} : \cdots : a_n]$$

$$\left[\frac{a_0}{a_k}, \ldots, \hat{1}, \ldots, \frac{a_n}{a_k}\right] \longleftarrow [\underline{a}]$$

At the polynomial level: $\mathbb{K}\left[\frac{x_0}{x_k}, \ldots, \hat{1}, \ldots, \frac{x_n}{x_k}\right]$ is the coordinate ring of $U_k \simeq \mathbb{A}^n$

① Given $W = V_{proj}(F_1, \ldots, F_m) \longrightarrow W \cap U_k = V(F_1^{(k)}, \ldots, F_m^{(k)})$

$F_i \in \mathbb{K}[x_0, \ldots, x_n]$ homogeneous

Affinization: $F_i \rightsquigarrow F_i^{(k)} = F_i\left(\frac{x_0}{x_k}, \ldots, 1, \ldots, \frac{x_n}{x_k}\right)$

② **Counterpart:**

$V = V(f_1, \ldots, f_s) \subseteq \mathbb{A}^n \simeq U_k \rightsquigarrow W = V_{proj}(f_1^h, \ldots, f_s^h) \subseteq \mathbb{P}^n$ is a projective variety with $W \cap U_k = V$.

Homogenization: $f \in \mathbb{K}[y_1, \ldots, y_n] \rightsquigarrow f^h(y_0, \ldots, y_n) = y_0^{\deg f} f\left(\frac{y_1}{y_0}, \ldots, \frac{y_n}{y_0}\right)$

Q1: Is W the smallest projective variety containing V? **A:** NOT in general!

- Homogenization & affinization are almost inverse operations:

Lemma 1: (1) Given $f \in \mathbb{K}[x_0, \ldots, \hat{x}_k, \ldots, x_n]$ we have $(f^h)^{(k)} = f$.

(2) Given $F \in \mathbb{K}[x_0, \ldots, x_n]$ homogeneous of degree d, we have $(F^{(k)})^h \mid F$ & equality holds if and only if $x_k \nmid F$. In general: $F = x_k^d (F^{(k)})^h$ with $d = \deg F - \deg f^{(k)}$.

Proof Easy exercise using the definitions.

Example: $F(x,y,z) = x^2 z + z^2 y$

$f^{(3)} = x^2 + y$ & $(f^{(3)})^h = x^2 + zy \neq F \rightsquigarrow F = z(f^{(3)})^h$.

Definition: Given an ideal $\mathfrak{a} \subseteq \mathbb{K}[x_1, \ldots, x_n]$, we define $\mathfrak{a}^h \subseteq \mathbb{K}[x_0, x_1, \ldots, x_n]$ as the homogeneous ideal $\mathfrak{a}^h = \langle f^h : f \in \mathfrak{a} \rangle$.

⚠ To compute $\mathfrak{a}^h$ it is **NOT** enough to homogenize the generators of $\mathfrak{a}$.

Example: $\mathfrak{a} = \langle z - y^2, w - y^3 \rangle \subseteq \mathbb{K}[y, z, w]$

$$J = \langle \underset{\substack{\parallel \\ zx - y^2}}{(z-y^2)^h}, \underset{\substack{\parallel \\ wx^2 - y^3}}{(w-y^3)^h} \rangle \subseteq \mathbb{K}[y, z, w]$$

Claim: $J \neq \mathfrak{a}^h$ since $g = f_2 - y f_1 = (w - y^3) - y(z - y^2) = w - yz \in \mathfrak{a}$

$g^h = xw - yz \in \mathfrak{a}^h$ but $g^h \notin J$.

($\deg f_1^h = 2$, $\deg f_2^h = 3$, deg $^h = 2$ & g^h is NOT a scalar multiple of f_2^h)

Note: If generators of $\mathfrak{a}$ are a special type of Gröbner basis (wrt a graded monomial order) then $\mathfrak{a}^h = \langle f_1^h, \ldots, f_s^h \rangle$.

Our next objectives are: ① confirm that all Zariski closed sets in $\mathbb{P}^n$ are projective varieties
② given $V \subseteq \mathbb{A}^n \simeq U_0 \subseteq \mathbb{P}^n$, compute $\overline{V} \subseteq \mathbb{P}^n$.

§1 Zariski topology on $\mathbb{P}^n$:

Theorem 1: Two equivalent definitions for the Zariski Topology.

① $W \subseteq \mathbb{P}^n$ closed $\iff$ $W \cap U_i \subseteq U_i \simeq \mathbb{A}^n_{\mathbb{K}}$ is Zariski closed $\forall i = 0, \ldots, n$

② closed sets are those of the form $V_{proj}(S)$ for a set S of homogeneous polynomials

Proof: We prove the double implication.

① $\Rightarrow$ ②: By Lemmas 6 & 7 §16.2 we know that $V_{proj}(S)$ are closed sets for a topology on $\mathbb{P}^n$. We check that they are closed for the Zariski Top.

We may assume $S = \langle f_1, \ldots, f_m \rangle$ where $f_1, \ldots, f_m$ are homogeneous polynomials.

$$V_{proj}(S) \cap U_k = \{[\underline{a}] \in \mathbb{P}^n \mid f_1(\underline{a}) = \cdots = f_m(\underline{a}) = 0 \ \& \ a_k \neq 0\}$$
$$= \{ \underline{a} = (\tfrac{a_0}{a_k}, \ldots, \tfrac{a_{k-1}}{a_k}, 1, \tfrac{a_{k+1}}{a_k}, \ldots, \tfrac{a_n}{a_k}) \mid f_1^{(k)}(\underline{a}) = \cdots = f_m^{(k)}(\underline{a}) = 0 \}$$
$$= V(\{f_1^{(k)}, \ldots, f_m^{(k)}\}) \subseteq U_k \simeq \mathbb{A}^n_{\mathbb{K}} \quad \text{is Zariski closed.}$$

Since this is true for all $k = 0, \ldots, n$, we conclude by ① that $V_{proj}(S)$ is Zariski closed in $\mathbb{P}^n$.

① ⇒ ②: We must show that the only sets satisfying ① are those of the form $V_{proj}(S)$ where S is a set of homogeneous polynomials.

The statement is clear for $W=\phi$ or $X=\mathbb{P}^n$. They are close in the Zariski Topology & $\phi = V_{proj}(\{1\})$, $\mathbb{P}^n = V_{proj}(\{0\})$

- Pick $\phi \subsetneq W \subsetneq \mathbb{P}^n$ Z. closed. By construction $W_k := W \cap U_k \subseteq U_k \simeq \mathbb{A}^n$ is closed, So $W_k = V(I^{(k)})$ for some $I^{(k)} \subseteq \mathbb{K}[\frac{x_0}{x_k}, \ldots, \widehat{\frac{x_k}{x_k}}, \ldots, \frac{x_n}{x_k}]$

Note: By construction, each $f \in I^{(k)}$ has the form $f = h(\frac{x_0}{x_k}, \ldots, 1, \ldots, \frac{x_n}{x_k})$. In particular, each monomial in f has total degree 0 in $x_0, \ldots, x_n$.

Hence, each $f \in I^{(k)}$ is a quotient of the form $f = \frac{g}{(x_k)^d}$ & $g \in \mathbb{K}[x_0, \ldots, x_n]$ is homogeneous of degree d.

Thus, we may write $W_k = V(\{\frac{g_{k,1}}{(x_k)^{d_{k,1}}}, \ldots, \frac{g_{k,r_k}}{(x_k)^{d_{k,r_k}}}\})$ for $\underset{\deg g_{k,1}}{\overset{\shortparallel}{d_{k,1}}}, \ldots, \underset{\deg g_{k,r_k}}{\overset{\shortparallel}{d_{k,r_k}}} \geq 0$.

& $g_{k,1}, \ldots, g_{k,r_k} \in \mathbb{K}[x_0, \ldots, x_n]$ are homogeneous with $x_k \nmid g_{k,j}$ $\forall j = 1, \ldots, r_k$.

- By repeating elements if necessary, we can assume $r_0 = r_1 = \cdots = r_n = r$.

- Set $I = \langle x_k g_{k,j} : k = 0, \ldots, n ; j = 1, \ldots, r \rangle \subseteq \mathbb{K}[x_0, \ldots, x_n]$. By construction I is a homogeneous ideal.

Claim: $W = V_{proj}(I)$

Pf/ ($\supseteq$) $V_{proj}(I) \cap U_k \subseteq V(\{f_{k,1}, \ldots, f_{k,r}\})$ $\forall k$ by construction of I, so $V_{proj}(I) \cap U_k \subseteq W_k$ $\forall k$. Thus,

$$V_{proj}(I) = \bigcup_{k=0}^{n} (V_{proj}(I) \cap U_k) \subseteq \bigcup_{k=0}^{n} W_k = W$$

($\subseteq$) For this inclusion, we need to worry about interactions between polynomials homogeneized from different affine patches. The proof will explain why we need to include the x_k factors:

Pick $[\underline{a}] \in W$. We need to check that each $x_k g_{k,j}$ vanishes on $[\underline{a}]$. $\forall j$.

(1) If $[\underline{a}] \in H_k = \mathbb{P}^n \setminus U_k$, then $[\underline{a}] \in V_{proj}(x_k)$ so $[\underline{a}] \in V_{proj}(x_k g_{k,j})$

(2) If $[\underline{a}] \in U_k$, then $[\underline{a}] \in W_k = V(g_{k,j})$ because $f_{k,j} = \frac{g_{k,j}}{x_k^{d_{k,j}}}$

These are the two possibilities, so we are done. $\square$

Remark: $J = \langle g_{k,j} : k=0,\dots,n \quad j=1,\dots,r\rangle$ will not work!

Example: $W = \{[1:0],[0:1]\} \subseteq \mathbb{P}^1$ is closed, since

$W \cap U_0 = \{[1:0]\} = V\left(\frac{y}{x}\right)$ & $W \cap U_1 = \{[0:1]\} = V\left(\frac{x}{y}\right)$ closed.

We set $g_{0,1} = y$ & $g_{1,1} = x$ $\qquad I = \langle xy, yx\rangle = \langle xy\rangle$

$W = V_{proj}(I)$ **BUT** $W \neq V_{proj}(\langle y, x\rangle) = V_{proj}(I^0) = \emptyset$.

Corollary 1: $\{\mathbb{P}^n \smallsetminus V_{proj}(S)\}_{S \subseteq \mathbb{K}[\underline{x}] \text{ homogeneous}}$ defines the Zariski topology on $\mathbb{P}^n$.

§2 Projective closures:

Q: How to compute the Zariski closure $\overline{V} \subseteq \mathbb{P}^n$ of $V \subseteq \mathbb{A}^n \cong U_0$?

Theorem 2: Given $V \subseteq \mathbb{A}^n \simeq U_0$, let $I = I(V)$. Then

(1) $\overline{V} = V_{proj}(I^h) \subseteq \mathbb{P}^n$

(2) If V is irreducible, so is $\overline{V} \subseteq \mathbb{P}^n$

(3) No irreducible component of $\overline{V}$ lies in the hyperplane $H_0 = V_{proj}(x_0) \subseteq \mathbb{P}^n$.

Proof: Set $W = V_{proj}(I^h)$. By Proposition 1, it is Zariski closed in $\mathbb{P}^n$

(1). By construction $V \subseteq W \cap U_0$ so $\overline{V} \subseteq W$.

. To finish, pick any projective variety Z with $Z \supseteq V$. We will show $Z \supseteq W$.

We write $Z = V_{proj}(F_1,\dots,F_s)$. Since F_i vanishes on Z, we have $F_i^{(0)} = f_i$ vanishes on $Z \cap U_0 \supseteq V$. Thus $f_i \in I(V) = I$ & $f_i^h \in I^h$. So f_i^h vanishes on W.

By construction, $F_i = x_0^{d_0} f_i^h$ for some $d_0 \geq 0$ so F_i vanished on W as well.

Conclusion: $W \subseteq V_{proj}(F_1,\dots,F_s) = Z$.

(2) Assume $\overline{V} = W_1 \cup W_2$ with $W_1, W_2 \subseteq \mathbb{P}^n$ Zariski closed

Then $V \subseteq \overline{V} \cap U_0 = (W_1 \cap U_0) \cup (W_2 \cap U_0)$ & $W_i \cap U_0$ is closed in U_0 $\forall i$.

Since V is irreducible, we have $V \subseteq W_1 \cap U_0$ or $V \subseteq W_2 \cap U_0$

But then $\overline{V} \subseteq \overline{W_i \cap U_0} \subseteq \overline{W_i} = W_i$ for $i = 1$ or 2.

Conclusion: $\overline{V}$ is irreducible.

(3) We write the irreducible decomposition of $\overline{V} = W$:

$$W = W_1 \cup \cdots \cup W_m \qquad (*)$$

where W_i is irreducible in $\mathbb{P}^n$ $\forall i = 1, \ldots, m$.

Suppose $W_1 \subseteq H_0$. Then

$$V \subseteq W \cap U_0 = (W_1 \cap \cdots \cap W_m) \cap U_0$$
$$= (W_2 \cup \cdots \cup W_m) \cap U_0.$$

This shows $Z = W_2 \cup \cdots \cup W_m$ is a projective variety containing V. By (1) we have $Z \supseteq \overline{V} = W$ so $W_2 \cup \cdots \cup W_m \supseteq W_1 \cup \cdots \cup W_m \supseteq W_1$.

This implies $W_1 \subseteq W_i$ for some i, so (*) was not the irreducible decomp. of W (there cannot be any redundancies!) $\square$

Remark: Main drawback of this statement is that we need to know $I(V)$! We can get away with any ideal $\mathfrak{a}$ defining V if $\overline{\mathbb{K}} = \mathbb{K}$.

Theorem 3: Assume $\overline{\mathbb{K}} = \mathbb{K}$ & let $\mathfrak{a} \subseteq \mathbb{K}[x_1, \ldots, x_n]$ be an ideal. Set $V = V(\mathfrak{a}) \subseteq \mathbb{A}^n$. Then $V_{proj}(\mathfrak{a}^h) = \overline{V}$ in $\mathbb{P}^n$ ($V \subseteq U_0 \subseteq \mathbb{P}^n$).

Proof: We use the Nullstellensatz for $W = V_{proj}(\mathfrak{a}^h)$.

By construction $W \cap U_0 \supseteq V$ & W is Z. closed in $\mathbb{P}^n$ so $W \supseteq \overline{V}$.

To prove the equality, we take any projective variety $Z \supseteq V$ & show $Z \supseteq W$.

Write $Z = V_{proj}(F_1, \ldots, F_r)$ & set $f_i = F_i^{(0)} = F_i(1, \frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0})$

Since f_i vanishes on V we get $\langle f_1, \ldots, f_r \rangle \subseteq I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$ by the Nullstellensatz. Thus $f_i^{m_i} \in \mathfrak{a}$ for some $m_i \geq 1$.

In particular $(f_i^{m_i})^h = (f_i^h)^{m_i} \in \mathfrak{a}^h$ for some $m_i \geq 1$.

Since $F_i = x_0^{d_i} f_i^h$ for some m_i, we conclude $F_i^{m_i} \in \mathfrak{a}^h \ \forall i$

Thus, $Z = V_{proj}(\{F_1, \dots, F_r\}) \supseteq V_{proj}(\sqrt{\mathfrak{a}^h}) = V_{proj}(\mathfrak{a}^h) = W$ $\square$

⚠ The statement fails if $\overline{\mathbb{K}} = \mathbb{K}$.

Example: Set $\mathbb{K} = \mathbb{R}$ & $\mathfrak{a} = \langle y^2 + z^4 \rangle \subseteq \mathbb{R}[y, z]$. $V(\mathfrak{a}) = \{(0,0)\}$

$V = V(\mathfrak{a}) \hookrightarrow \mathbb{P}^2$ via $V = \{(1:0:0)\}$.

Then $\mathfrak{a}^h = \langle x^2y^2 + z^4 \rangle$ & $V_{proj}(\mathfrak{a}^h) = \{(1:0:0)\} \cup \{(0:1:0)\}$

$(V_{proj}(\mathfrak{a}^h) \cap U_0 = V$, $V_{proj}(\mathfrak{a}^h) \cap U_1 = V\left(\left(\frac{x}{y}\right)^2 + \left(\frac{z}{y}\right)^4\right) = \{(0,0)\} \in U_1$,

$V_{proj}(\mathfrak{a}^h) \cap U_2 = V\left(\left(\frac{x}{z}\right)^2\left(\frac{y}{z}\right)^2 + 1\right) = \phi$ in $\mathbb{R}^2$.