Lecture XX: Projective Morphisms

Up to this point, we discussed the objects in the category of projective varieties.
Next Task: Describe maps in the category.

§1 Function theory on projective varieties:

• Recall (Lectures 10, 11 & 12):
Given $V \subseteq \mathbb{A}^n_{\mathbb{K}}$, $W \subseteq \mathbb{A}^m_{\mathbb{K}}$, we had several types of maps $\varphi: V \longrightarrow W$

(1) Morphisms: $\varphi = (f_1, \ldots, f_m)$ $\quad f_i \in \mathbb{K}[V]$ with $\varphi(V) \subseteq W$.

$$\mathrm{Mor}(V,W) = \mathrm{Hom}_{\mathbb{K}\text{-alg}}(\mathbb{K}[W], \mathbb{K}[V])$$

(2) Rational maps: φ defined on a dense open U of V & $\varphi = \left(\frac{g_1}{h_1}, \ldots, \frac{g_m}{h_m}\right)$
with $g_1, \ldots, g_m, h_1, \ldots, h_m \in \mathbb{K}[V]$ $\quad h_1, \ldots, h_m$ nowhere 0 on U & $\varphi(U) \subseteq W$
(so $U \subseteq D(h_1 \cdots h_m)$)

function fields

If V, W are irreducible, $\mathrm{Rat}(V,W) = \mathrm{Hom}_{\mathbb{K}\text{-alg}}(\mathbb{K}(W), \mathbb{K}(V))$

(3) Regular maps: V, W irreducible & given any $p \in V$ $\exists\, U$ open with $p \in U \subseteq V$,
$g_1, \ldots, g_m, h_1, \ldots, h_m \in \mathbb{K}[U]$ h_i nowhere 0 on U $\forall i$ with $\varphi|_U = \left(\frac{g_1}{h_1}, \ldots, \frac{g_m}{h_m}\right)$

• Since $\mathbb{P}^n = \bigcup_{k=0}^{n} U_k$ & $U_k \simeq \mathbb{A}^n_{\mathbb{K}}$ for each k, we can use the affine definitions to determine morphisms / rational / regular maps on $\mathbb{P}^n$ & projective subvarieties of $\mathbb{P}^n$. More precisely:

STEP 1: Define these notions by using the affine condition on each $V \cap U_k \subseteq U_k \simeq \mathbb{A}^n_{\mathbb{K}}$ for $k = 0, \ldots, n$.

STEP 2: Check agreement on overlaps $V \cap U_k \cap U_j = (V \cap U_k) \cap (V \cap U_j)$ by using the transition functions $U_{jk} = U_j \cap U_k \longrightarrow U_{kj} = U_k \cap U_j$ & the homomorphisms

$$\mathbb{K}(U_{kj}) = \mathbb{K}\left(\frac{x_0}{x_k}, \ldots, \hat{1}, \ldots, \frac{x_n}{x_k}\right) \xrightarrow{\varphi^*_{kj}} \mathbb{K}\left(\frac{x_0}{x_j}, \ldots, \hat{1}, \ldots, \frac{x_n}{x_j}\right) = \mathbb{K}(U_{jk})$$

$$i \neq j \qquad \frac{x_i}{x_k} \longmapsto \left(\frac{x_i}{x_j}\right) \Big/ \left(\frac{x_k}{x_j}\right)$$

$$x_j / x_k \longmapsto (x_k / x_j)^{-1}$$

This will be the approach for abstract affine varieties (similar constructions are used for Manifolds). For $V \subseteq \mathbb{P}^n$, we can bypass this and work with $S(V)$.

Remark: We saw that elements of $S(V)$ do not define maps $V \longrightarrow \mathbb{A}^1_{\mathbb{K}}$, but degree 0 ratios of homogeneous elements of $S(V)$ do!

- Assume $V \subseteq \mathbb{P}^n$ is irreducible & $\mathbb{K}$ is infinite, so $S(V)$ is a graded domain.

Proposition 1: For each k: $V \cap U_k$ is either $\emptyset$ or it is dense in V. Furthermore, if $V \cap U_k \neq \emptyset$, it is irreducible in $U_k \simeq \mathbb{A}^n$.

Proof: For each k, the set $V_k = V \cap U_k$ is open in V. Since

$$V = (V \cap U_k) \cup (V \cap H_k), \quad \text{and} \quad \overline{V \cap U_k} \subseteq V \quad \text{we get}$$

$$V = \overline{V \cap U_k} \cup (V \cap H_k)$$

Since V is irreducible, either $V \cap H_k = V$ (ie $V \cap U_k = \emptyset$) or $\overline{V \cap U_k} = V$ (ie, $V \cap U_k$ is dense in V). In the latter case, Theorem 2 §19.2 confirms $V \cap U_k$ must be irreducible in U_k (If $V \cap U_k = W_1 \cup \cdots \cup W_r$ is the irred. decomp of $V \cap U_k$, then $V = \overline{V \cap U_k} = \overline{W_1} \cup \cdots \cup \overline{W_r}$ is the irred decomp of V, so $r=1$) $\square$

Definition: The field of rational functions on $V \subseteq \mathbb{P}^n$ irreducible is

$$\mathbb{K}(V) = \left\{ \frac{\overline{F}}{\overline{G}} \mid \overline{F}, \overline{G} \in S(V)_d \text{ for some } d \ \& \ \overline{G} \neq 0 \right\} / \sim$$

where

$$\frac{\overline{F_1}}{\overline{G_1}} \sim \frac{\overline{F_2}}{\overline{G_2}} \iff \overline{F_1}\,\overline{G_2} - \overline{F_2}\,\overline{G_1} = 0 \text{ in } S(V) \qquad \left(\text{ie } F_1G_2 - F_2G_1\big|_V = 0\right)$$

$$\iff F_1G_2 - F_2G_1 \in I^h(V).$$

Note that $\sim$ is independent of the representatives $F, G \in S(V)_d$ that we pick.

Lemma 1: $\sim$ is an equivalence relation.

Proof: - The reflexive and symmetric properties are clear ($S(V)$ is a $\mathbb{K}$-algebra)

- Transitivity will follow from the fact that $S(V)$ is a domain. Indeed, assume

$$\overline{F_1}/\overline{G_1} \sim \overline{F_2}/\overline{G_2} \iff F_1G_2 - F_2G_1 \in I^h(V). \qquad (\text{Say } \overline{F_1} \in S(V)_{d_1})$$

$$\overline{F_2}/\overline{G_2} \sim \overline{F_3}/\overline{G_3} \iff F_2G_3 - F_3G_2 \in I^h(V) \qquad (\text{Say } \overline{F_2} \in S(V)_{d_2})$$

To show: $F_1G_3 - G_1F_3 \in I^h(V)$

Since $\overline{G_2} \neq 0$ this means $G_2 \notin I^h(V)$.

Thus $\overline{G_2(F_1G_3 - G_1F_3)} = \overline{F_1G_2G_3 - G_1G_2F_3} = \overline{F_1G_2}\ \overline{G_3} - \overline{G_2F_3}\ \overline{G_1}$
$$= \overline{F_2G_1G_3} - \overline{F_2G_3G_1} = \overline{0} = 0 \text{ in } S(V)$$

Since $S(V)$ is a domain & $\overline{G_2} \neq 0$ in $S(V)$ we conclude that $\overline{F_1}\,\overline{G_3} - \overline{G_1}\,\overline{F_3} = 0$ in $S(V)$, ie $\overline{F_1}/\overline{G_1} \sim \overline{F_3}/\overline{G_3}$.

Lemma 2: $\mathbb{K}(V)$ is a field extension of $\mathbb{K}$. Moreover, $\operatorname{trdeg}(\mathbb{K}(V)|\mathbb{K}) \leq n$.

Proof: The field structure is inherited from the structure of $S(V)$ & the grading.

For the degree statement, we pick k with $V \cap U_k \neq \emptyset$.

Claim: $\mathbb{K}(V) = \mathbb{K}\left(\frac{\overline{x_0}}{\overline{x_k}}, \ldots, \frac{\overline{x_n}}{\overline{x_k}}\right)$

Pf/ Any element of $\mathbb{K}(V)$ is of the form $\frac{\overline{F}}{\overline{G}}$ where $F, G \in \mathbb{K}[x_0 \ldots x_n]_d$ for some d with $G \notin I^h(V)$

- Since V is irreducible, then $V \cap U_k$ is dense in V by Proposition 1. In particular, G cannot vanish everywhere on $V \cap U_k$, so $x_k \nmid G$.

Lemma 1 §19

$$\text{Thus, } \frac{\overline{F}}{\overline{G}} = \frac{\overline{x_k}^d\, F^{(k)}\left(\frac{\overline{x_0}}{\overline{x_k}}, \ldots, \frac{\overline{x_n}}{\overline{x_k}}\right)}{\overline{x_k}^d\, G^{(k)}\left(\frac{\overline{x_0}}{\overline{x_k}}, \ldots, \frac{\overline{x_n}}{\overline{x_k}}\right)} = \frac{F^{(k)}}{G^{(k)}\left(\frac{\overline{x_0}}{\overline{x_k}}, \ldots \frac{\overline{x_n}}{\overline{x_k}}\right)} \in \mathbb{K}\left(\frac{\overline{x_0}}{\overline{x_k}}, \ldots \frac{\overline{x_n}}{\overline{x_k}}\right) \quad (*)$$

- Conversely, $\frac{\overline{x_i}}{\overline{x_k}} \in \mathbb{K}(V)$ $\forall i \neq k$ so $\mathbb{K}\left(\frac{\overline{x_0}}{\overline{x_k}}, \ldots, \frac{\overline{x_n}}{\overline{x_k}}\right) \subseteq \mathbb{K}(V)$. □

By construction $\mathbb{K}(V)$ & $\mathbb{K}\left(\frac{\overline{x_0}}{\overline{x_k}}, \ldots, \frac{\overline{x_n}}{\overline{x_k}}\right) \supseteq \mathbb{K}$ share the same operations

Corollary 1: Fix $V \subseteq \mathbb{P}^n$ irreducible projective variety with $V \cap U_0 \neq \emptyset$. Then $\mathbb{K}(V \cap U_0) \cong \mathbb{K}(V)$ as isomorphic rings

Proof: The function field of $\mathbb{K}(V \cap U_0)$ is $\mathbb{K}(V \cap U_0) = \mathbb{K}\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right)$

Remark: We'll use the transcendence degree of $\mathbb{K}(V)|\mathbb{K}$ to define the dimension of $V \subseteq \mathbb{P}^n$.

Corollary 2: $\{\text{Rational functions } \underset{V \subseteq \mathbb{P}^n \text{ irred}}{V \dashrightarrow \mathbb{A}^1_{\mathbb{K}}}\} \longleftrightarrow \mathbb{K}(V)$

Proof: Rational functions are determined on each $V \cap U_k \subseteq U_k$ with $V \cap U_k \neq \emptyset$. The compatibility on overlaps $V_{ij} = (V\cap U_i) \cap (V \cap U_j) \xrightarrow{\varphi_{ij}} V_{ji} = (V\cap U_j) \cap (V \cap U_i)$, where $V_{ij} \subseteq U_i$ and $V_{ji} \subseteq U_j$,

comes from $(*)$: Indeed, ratios $\frac{f}{g}$ on $\mathbb{K}(U_k \cap V)$ correspond to $\frac{F}{G}$ where $F = x_k^{d-\deg f} f^h$ & $G = x_k^{d-\deg g} g^h$ for $d \gg 0$. This will certify that the assignment on $(V \cap U_k)$'s agree on overlaps, meaning it is compatible with the transition functions. $\square$

§2. Regular maps on $\mathbb{P}^n$:

Fix $V \subseteq \mathbb{P}^n$ irreducible projective variety. Our next goal is to define $\mathcal{O}_V$, the sheaf of regular functions on V.

Definition: Fix $U \subseteq V$ open set & $\varphi: U \longrightarrow \mathbb{A}^1_{\mathbb{K}}$

(1) Given $p \in U$ we say φ is <u>regular at p</u> if $\exists\, \tilde{U}$ open $p \in \tilde{U} \subseteq U$ such that $\varphi|_{\tilde{U}} \in \mathbb{K}(U)$. Equivalently, if $p \in U_k$, then $\varphi|_{U\cap U_k}: U \cap U_k \longrightarrow \mathbb{A}^1_{\mathbb{K}}$ is a regular function at p on the open subset $U \cap U_k$ of $V \cap U_k \subseteq U_k \simeq \mathbb{A}^n$.

(2) We say φ is <u>regular</u> if it is regular at each point $p \in U$. Equivalently, φ must be locally regular, ie for each $k=0,\dots,n$ the function $\varphi_k : U \cap U_k \longrightarrow \mathbb{A}^1_{\mathbb{K}}$ is a regular function on $U \cap U_k \subseteq U_k \simeq \mathbb{A}^n_{\mathbb{K}}$

Lemma 3: Regular maps are continuous with respect to the Zariski Topology

Proof: Continuity is a local condition & regular maps on affine varieties are continuous.

Definition: If $V \subseteq \mathbb{P}^n$ & $W \subseteq \mathbb{P}^m$ are irreducible projective varieties, a map $\varphi: V \longrightarrow W$ is <u>regular</u> if and only if $\varphi^*(f): V \longrightarrow \mathbb{A}^1_{\mathbb{K}}$ is regular for all regular maps $f: W \longrightarrow \mathbb{A}^1_{\mathbb{K}}$.

Remark: Equivalently, φ must be continuous & locally regular. More precisely, for each $k=0,\dots,m$, $\varphi^{-1}(U_k)$ is open in $V\subseteq\mathbb{P}^n$ & for each $j=0,\dots,n$ the maps

$$\varphi_{kj}: \underbrace{V\cap\varphi^{-1}(U_k)\cap U_j}_{\subseteq \mathbb{A}^n_{\mathbb{K}}} \longrightarrow \underbrace{W\cap U_k}_{\subseteq \mathbb{A}^m_{\mathbb{K}}}$$

are regular maps between affine varieties.

Definition: For $V\subseteq\mathbb{P}^n$ irreducible, we define $\mathcal{O}_V$ to be the sheaf of regular functions on V, with the standard restrictions. More precisely, for each $U\subseteq V$ open we have

$$U \longmapsto \mathcal{O}_V(U) = \{\varphi: U\to\mathbb{A}^1_{\mathbb{K}} \text{ regular at } U\}$$

The regular functions at a point $p\in V$ correspond to the stalk $\mathcal{O}_{V,p}$.

Remark: This definition can be recovered as the gluing of the sheaves on the affine varieties $(V\cap U_k)_{k=0}^n$ where $V\cap U_k\subseteq U_k\simeq\mathbb{A}^n_{\mathbb{K}}$ & $\mathbb{K}[\mathbb{A}^n_{\mathbb{K}}]=\mathbb{K}\left[\frac{x_0}{x_k},\dots,\hat{1},\dots,\frac{x_n}{x_k}\right]$, given the isomorphisms $\mathcal{O}_{V\cap U_k}\big|_{U_j\cap V} \overset{\varphi_{jk}}{\simeq} \mathcal{O}_{V\cap U_j}\big|_{U_k\cap V}$ induced by the transition functions $U_{kj}\longrightarrow U_{jk}$. This general gluing procedure is described in Problem 20 HW4.

⚠ As with the affine case, regular maps are NOT rational.

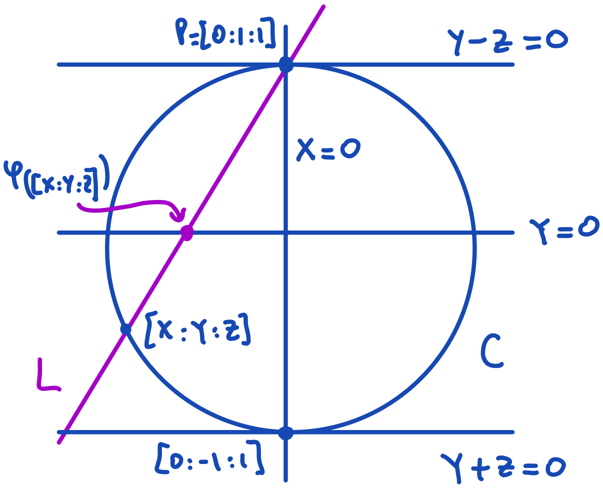
Example: Fix $V=V_{proj}(x^2+y^2-z^2)\subseteq\mathbb{P}^2$ & $\varphi: V \dashrightarrow \mathbb{P}^1$ rational

$$[x:y:z]\longmapsto[x:z-y]$$

This map is well-defined outside $x=z-y=0$ ie $p=[0:1:1]\in V$.

- On U_0: $\varphi_0: V\cap U_1 \longrightarrow \mathbb{P}^1$
  $$\left(\tfrac{y}{x},\tfrac{z}{x}\right)\longmapsto\left[1:\tfrac{z}{x}-\tfrac{y}{x}\right]$$
- On U_1: $\varphi_2: V\cap U_2 \dashrightarrow \mathbb{P}^1$ (issue on $p=(0,1)$)
  $$\left(\tfrac{x}{y},\tfrac{z}{y}\right)\longmapsto\left[\tfrac{x}{y}:\tfrac{z}{y}-1\right]$$
- On U_2: $\varphi_3: V\cap U_3 \dashrightarrow \mathbb{P}^1$ (issue on $p=(0,1)$)
  $$\left(\tfrac{x}{z},\tfrac{y}{z}\right)\longmapsto\left[\tfrac{x}{z},1-\tfrac{y}{z}\right]$$

Geometrically, φ corresponds to a stereographic projection from $P=[0:1:1]$



- Line L through $[x:y:z]$ & $[0:1:1]$ is $[\lambda x : \lambda y+\mu : \lambda z+\mu]$ $([\lambda:\mu]\in\mathbb{P}^1)$
- If $\lambda y+\mu=0$, then $\mu=-\lambda y$ & we get $[\lambda x : 0 : \lambda z-\lambda y] = [\lambda x : \lambda(z-y)]$ this point is $[x:z-y]$ ($\lambda=0 \Rightarrow \mu=0$, which is not allowed)

- We can extend φ to all V by defining $\varphi(p)=[1:0]\in U_0$.

Claim: φ is regular at p.

Pf/ Use coordinates $[S:T]$ on $\mathbb{P}^1$ & analyze $\varphi^{-1}(U_0) \longrightarrow U_0 \simeq \mathbb{A}^1_{\mathbb{K}}$ (coord $=\frac{T}{S}$)
$\varphi^{-1}(U_1) \longrightarrow U_1 \simeq \mathbb{A}^1_{\mathbb{K}}$ (coord $=\frac{S}{T}$)

(1) $\varphi^{-1}(U_0) = (V \setminus \{[x:y:z] \mid x=0\}) \cup \{p\} = V \setminus \{[0:-1:1]\}$

Reason: If $z-y=0$ or $z+y=0$, then $y^2-z^2=0$ & $x^2+y^2-z^2=0$ forces $x=0$

- On $\varphi^{-1}(U_0)\setminus\{p\}$, the map becomes $[x:y:z] \longrightarrow \frac{z-y}{x} \in U_0 \simeq \mathbb{A}^1$. This is a rational function defined everywhere on $\varphi^{-1}(U_0)\setminus\{p\}$.

Since on V we have $x^2=(z-y)(z+y)$ we get $\frac{z-y}{x} = \frac{x}{z+y}$

This expression is valid on a neighborhood of p & $\frac{x}{z+y}(p)=0$ as expected from $\varphi(p)=[1:0]$

(2) $\varphi^{-1}(U_1) = V \setminus (\{[x:y:z] \mid z-y=0\}) = V \setminus \{[0:1:1]\}$

On $\varphi^{-1}(U_1)$, the map becomes $[x:y:z] \longrightarrow \frac{x}{z-y}$ this is a rational function, defined everywhere on $\varphi^{-1}(U_1)$.

Conclusion: φ is a regular function on V, but it is not a rational function on V.