Lecture XXI: Projective morphisms II

Recall: Last time we defined rational & regular maps to $\mathbb{A}^1_{\mathbb{K}}$ from any $V \subseteq \mathbb{P}^n$ irreducible projective variety by gluing the notions for the restrictions to all $V \cap U_k \subseteq U_k \simeq \mathbb{A}^n$

- Rational maps to $\mathbb{A}^1_{\mathbb{K}}$ ($\mathbb{K}$ infinite) $\longleftrightarrow$ $\mathbb{K}(V) = \{ \frac{\overline{F}}{\overline{G}} : \overline{F}, \overline{G} \in S(V)_d \quad \overline{G} \neq 0 \}/_\sim \cong \mathbb{K}(V \cap U_k)$ for any k with $V \cap U_k \neq \emptyset$.

§1 Rational & regular maps to $\mathbb{P}^m$:

- Regular maps allow us to define isomorphisms between projective varieties:

Definition. - A regular map $\varphi: V \longrightarrow W$ between irreducible projective varieties is an isomorphism if $\exists\ \psi: W \longrightarrow V$ regular with $\psi \circ \varphi = id_V$ & $\varphi \circ \psi = id_W$

- Two irreducible projective varieties are isomorphic if there exists a regular isomorphism between them.

Q: Fix $V \subseteq \mathbb{P}^n$ irreducible. What do regular maps $\varphi: V \longrightarrow \mathbb{P}^m$ look like?

Example: (last time) $V = V_{proj}(x^2 + y^2 - z^2) \subseteq \mathbb{P}^1$, $V \ni P = [0:1:1]$

$\varphi: V \dashrightarrow \mathbb{P}^1$, $[x:y:z] \longrightarrow [x: z-y]$ is well-defined on $V \smallsetminus P$

- φ extends to a regular map on V via $\varphi(P) = [1:0]$.

$\varphi^{-1}(U_0) = V \smallsetminus \{[0:-1:1]\}$, $\varphi^{-1}(U_1) = V \smallsetminus \{[0:1:1]\}$ $\qquad \mathbb{P}^1 = U_0 \cup U_1$

$$\varphi|_{\varphi^{-1}(U_0)}[x:y:z] = \frac{z-y}{x} = \frac{x}{z+y} \quad \& \quad P \in \varphi^{-1}(U_0) \qquad (x^2 = (z-y)(z+y) \text{ on } V)$$

$$\varphi|_{\varphi^{-1}(U_1)}[x:y:z] = \frac{x}{z-y}$$

- **Q:** How to build regular maps $\varphi: V \longrightarrow \mathbb{P}^m$ for $V \subseteq \mathbb{P}^n$ projective variety?
- **A:** In practice, pick $m+1$ homogeneous polynomials $G_0, G_1, \ldots, G_m$ of the same degree (not all 0 on V) and set $\varphi = [G_0 : G_1 : \cdots : G_m]$

Issue: If $V_{proj}(G_0, G_1, \ldots, G_m) \cap V \neq \emptyset$ we need to see if the definition on $V \smallsetminus V_{proj}(G_0, G_1, \ldots, G_m)$ can be extended to V. This will depend on V. In most cases, we'll only get a rational map on V.

Proposition 1: A rational map $\varphi: \mathbb{P}^n_{\mathbb{K}} \dashrightarrow \mathbb{P}^m_{\mathbb{K}}$ corresponds to a morphism $\tilde{\varphi}: \mathbb{A}^{n+1}_{\mathbb{K}} \longrightarrow \mathbb{A}^{m+1}_{\mathbb{K}}$ given by $\tilde{\varphi} = (F_1, \ldots, F_m)$ where $F_1, \ldots, F_m$ are homogeneous polynomials in $n+1$ variables of the same degree.

Proof: Since we are mapping to $\mathbb{P}^m_{\mathbb{K}}$, we can clear denominators.

- On an irreducible proj variety $V \subseteq \mathbb{P}^n_{\mathbb{K}}$ we get a similar rational result.

Proposition 2: If $V \subseteq \mathbb{P}^n_{\mathbb{K}}$ is an irreducible projective variety, a tuple $(F_0, \ldots, F_m)$ of homogeneous polynomials of the same degree, not all of which lie in $I^h(V)$ determine a rational map $V \overset{\varphi}{\dashrightarrow} \mathbb{P}^m_{\mathbb{K}}$. Furthermore, two such pairs $(F_i), (G_j)$ determine the same rational map if $F_iG_j - F_jG_i \in I^h(V)$ for all i, j.

Proof: On each standard open of $\mathbb{P}^n_{\mathbb{K}}$ we get $\varphi_{|\varphi^{-1}(U_i)}: V \cap \varphi^{-1}(U_i) \dashrightarrow U_i \simeq \mathbb{A}^n$ with $\varphi_{|\varphi^{-1}(U_i)} = \left(\frac{F_0}{F_i}, \ldots, \widehat{\frac{F_i}{F_i}}, \ldots, \frac{F_m}{F_i}\right)$. This is a rational map to $\mathbb{A}^n$.

For the second part, we notice that both tuples define the same map $\varphi: V \dashrightarrow \mathbb{P}^m$ iff $\forall i: \frac{F_i}{F_j} = \frac{G_i}{G_j}$ on $V \cap \varphi^{-1}(U_j)$ $\forall j$ $\Leftrightarrow$ $F_iG_j - G_iF_j \in I^h(V)$ $\forall i, j$.

$\Leftrightarrow$ $\frac{F_i}{F_j} = \frac{G_i}{G_j}$ in $K(V)$

Remark: Rational maps $V \overset{\varphi}{\dashrightarrow} W$ with $V \subseteq \mathbb{P}^n$, $W \subseteq \mathbb{P}^m$ irreducible coming from $\{F_0, \ldots, F_m\} \subseteq S(V)_d$, not all 0, requires $\varphi(v) \in W$ $\forall v \in V \setminus V_{proj}(F_0, \ldots, F_m)$

§2 Examples:

- Next, we give some examples of rational / regular maps $\mathbb{P}^n \dashrightarrow \mathbb{P}^m$

Example 1 Linear maps $\varphi_A: \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ correspond to matrices A of size $(m+1) \times (n+1)$. up to global scalar

If A has rank $n+1 \leq m+1$, then φ_A is a morphism. Otherwise, the coordinates have a common projective vanishing loci, so φ is only rational on $\mathbb{P}^n$

Definition: $\mathbb{P}GL_n = \{$linear isomorphisms $\mathbb{P}^n \longrightarrow \mathbb{P}^n\}$

- We have an additional notion arising from $\mathbb{P}GL_n$: projective equivalence!

Definition: Two varieties $V, V' \subseteq \mathbb{P}^n$ are projectively equivalent if there is an automorphism $A \in \mathbb{P}GL_{n+1}(\mathbb{K})$ of $\mathbb{P}^n$ carrying V onto V'. Equivalently iff (see HW5) $S(V)$ & $S(V')$ are isomorphic as graded $\mathbb{K}$-algebras.

Remark: Projectively equivalent $\Longrightarrow$ isomorphic, but not conversely.

Example ②: (Projection from a point) Consider $P = [0:0:1] \in \mathbb{P}^2$ & set

$$\pi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1 = H_2 \qquad \pi([x_0 : x_1 : x_2]) = [x_0 : x_1].$$

This map is not defined at P. It is rational, not defined on P.
Restricting π to a curve $V = V_{proj}(F)$ where F is homogeneous of degree d & $[0:0:1] \notin V$ we get a regular map $\varphi = \pi_{|V} : V \longrightarrow \mathbb{P}^1$.

Remark: If $\mathbb{K} = \overline{\mathbb{K}}$ & $[a:b] \in \mathbb{P}^1$ is generic, then $\varphi^{-1}([a:b])$ consists of d distinct points. (This will be a consequence of Bézout's Theorem)

⚠ Rational maps $\mathbb{P}^n \dashrightarrow \mathbb{P}^m$ can restrict to regular maps on subvarieties, even if they meet the locus where the map is not defined.

Example ③ $V = V_{proj}(xz - y^2) \subseteq \mathbb{P}^2$ & take $\pi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1 = H_2$ the projection from $P = [0:0:1]$ from Example ②.

Claim: $\pi_{|V} : V \longrightarrow \mathbb{P}^1$ is a regular map because (x_0, x_1) & (x_1, x_2) determine the same rational map to $\mathbb{P}^1$ on V. $(x_0 \cdot x_2 - x_1 \cdot x_1 \in I^h(V))$

A similar proof will yield the following statement (see HW5)

Lemma 1: Assume $\mathbb{K} = \overline{\mathbb{K}}$ & fix $V = V_{proj}(F) \subseteq \mathbb{P}^2$ for F homogeneous of degree 2. If F is irreducible, then V is isomorphic to $\mathbb{P}^1$.

Example ④ The Veronese map

Definition: Given $d, n \geq 1$ we define the Veronese map of degree d on $\mathbb{P}^n$ as

$$\nu_d : \mathbb{P}^n \longrightarrow \mathbb{P}^{\binom{n+d}{d}-1} \qquad [\underline{x}] \longmapsto [\underline{x}^I]_I$$

where $x^I = x_{i_0} \cdots x_{i_n}$ for $I = \{i_0, \dots, i_d\} \subseteq \{0, \dots, n\}$ (I multiset of size d)

The coordinates of $\nu_d([x])$ correspond to all monomials of degree d in $n+1$ variables.
This number is $\binom{n+d}{d}$.

Example: $\nu_2 : \mathbb{P}^1 \longrightarrow \mathbb{P}^{\binom{3}{2}-1} = \mathbb{P}^2$ $\qquad \nu_2(\mathbb{P}^1) = V_{proj}(xz - y^2) \subseteq \mathbb{P}^2.$

$$[x_0 : x_1] \longmapsto [x_0^2 : x_0x_1 : x_1^2]$$

Some observations are in order:

(1) ν_2 is injective ($\frac{u_1}{u_0}$ gives $\frac{x_1}{x_0}$, $\frac{u_2}{u_1}$ gives $\frac{x_0}{x_1}$ & one of these is nonzero)

(2) $\nu_2(\mathbb{P}^1) = V_{proj}(u_0u_2 - u_1^2) =: V \subseteq \mathbb{P}^2$

($\subseteq$) is clear

For ($\supseteq$) we need the following results:

Lemma 2: $V \subseteq U_0 \cup U_2$. $\qquad$ ($U_i \subseteq \mathbb{P}^2$ standard affine patch)

Proof: If $u_0u_2 = u_1^2$ then either $u_0 \neq 0$ or $u_2 \neq 0$ (o/w $[\underline{u}] = [0\,0\,0] \notin \mathbb{P}^2$)

Lemma 3: There exists a regular map $\tau : V \longrightarrow \mathbb{P}^1$ with $\tau = \nu_2^{-1}$.

Proof: It's enough to define it on U_0 & U_2 & check it agrees on $U_0 \cap U_2$

- $\tau_2^{(0)} := \tau|_{U_0} : U_0 \longrightarrow \mathbb{P}^1$, $\quad [u_0 : u_1 : u_2] \longmapsto [u_0 : u_1]$ $\qquad$ $\tau_2^{(2)} := \tau|_{U_2} : U_2 \longrightarrow \mathbb{P}^1$, $\quad [u_0 : u_1 : u_2] \longmapsto [u_1 : u_2]$

$\tau_2^{(0)}|_{U_0 \cap U_2} = \tau_2^{(2)}|_{U_0 \cap U_2}$ since $[u_0 : u_1] = [u_1 : u_2]$ if $u_0, u_2 \neq 0$ & $u_0u_2 = u_1^2$

Indeed, $(u_0, u_1) = \frac{u_0}{u_1}(u_1, u_2)$ & $\frac{u_0}{u_1} \neq 0$ on $U_2 \cap U_0$. ($u_1 \neq 0$ on $U_2 \cap U_0$)

Conclude: τ_2 is a regular map on V

Proposition 3: (1) $\tau_2 \circ \nu_2 = id_{\mathbb{P}^1}$ & (2) $\nu_2 \circ \tau_2 = id_V$

Proof: (1). $\tau_2 \circ \nu_2|_{U_0}([x_0 : x_1]) = \tau_2^{(0)}(\underbrace{[x_0^2 : x_0x_1 : x_1^2]}_{\in U_0 \subseteq \mathbb{P}^2}) = [x_0^2 : x_0x_1] \underset{x_0 \neq 0}{=} [x_0 : x_1]$ $\qquad$ ($U_0 \subseteq \mathbb{P}^1$)

- $\tau_2 \circ \nu_2|_{U_1}([x_0 : x_1]) = \tau_2^{(2)}(\underbrace{[x_0^2 : x_0x_1 : x_1^2]}_{\in U_1 \subseteq \mathbb{P}^2}) = [x_0x_1 : x_1^2] \underset{x_1 \neq 0}{=} [x_0 : x_1]$ $\qquad$ ($U_1 \subseteq \mathbb{P}^1$)

(2) • $\nu_2 \circ \tau_2|_{U_0}([u_0:u_1:u_2]) = \nu_2([u_0:u_1]) = [u_0^2 : u_0u_1 : u_1^2] = [u_0^2 : u_0u_1 : u_0u_2]$

$(u_0u_2 = u_1^2)$ in V

$\underset{u_0 \neq 0}{=} [u_0 : u_1 : u_2]$

• $\nu_2 \circ \tau_2|_{U_2}([u_0:u_1:u_2]) = \nu_2([u_1:u_2]) = [u_1^2 : u_1u_2 : u_2^2] \underset{u_1^2 = u_0u_2}{=} [u_0u_2 : u_1u_2 : u_2^2]$

$\underset{u_2 \neq 0}{=} [u_0 : u_1 : u_2]$.

Corollary 1: $V = \nu_2(\mathbb{P}^1)$

Proof: Proposition 3 (2) says $\underset{V}{\underset{\Uparrow}{v}} = \nu_2(\tau_2(v))$ so $v \in \nu_2(\mathbb{P}^1)$.

So $V \subseteq \nu_2(\mathbb{P}^1)$. The other inclusion was already known. $\square$

The same ideas will work for $\nu_d : \mathbb{P}^1 \longrightarrow \mathbb{P}^d$.