Lecture XXII: Projective Morphisms III

Last Time, we showed that regular & rational maps $\varphi: \mathbb{P}^n \longrightarrow \mathbb{P}^m$ correspond to tuples of m+1 homogeneous polynomials of the same degree.

Restricting these maps to irreducible projective varieties $V \subseteq \mathbb{P}^n$ produces rational / regular map via a tuple of (m+1) elements in $S(V)_d$ for some $d \geq 0$.

⚠ Not all rational / regular maps on $V \subseteq \mathbb{P}^n$ are of this form! (we'll see an example next time)

- Regular isomorphisms $\varphi: V \longrightarrow W$ = regular maps that are invertible & their inverse is also a regular map

Q: What happens for rational maps?

As in the affine case, we can reduce to $\varphi: V \dashrightarrow W$ where $V \subseteq \mathbb{P}^n$, $W \subseteq \mathbb{P}^m$ are irreducible & φ is dominant (i.e., image of φ is dense in W). This will allow us to compose rational functions (domain of rational functions on irred varieties are open & non-empty, hence they are dense)

Definition: Given V, W irreducible projective varieties & $\varphi: V \dashrightarrow W$ rational, we say φ is birational if $\exists\ \psi: W \dashrightarrow V$ rational with $\psi \circ \varphi = id_V$ & $\varphi \circ \psi = id_W$.

We say that two irreducible projective varieties V & W are birational to each other or birationally isomorphic if $\exists\ \varphi: V \dashrightarrow W$ birational map.

Example: $\varphi: \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ $\quad \varphi([x_0:x_1:x_2]) = [x_1x_2 : x_0x_2 : x_0x_1] = [\frac{1}{x_0} : \frac{1}{x_1} : \frac{1}{x_2}]$

is an example of a birational map of $\mathbb{P}^2$. It's called a Cremona transformation.

Fun fact: This map sends lines in $\mathbb{P}^2$ to conics in $\mathbb{P}^2$.

Theorem 1: Fix $V \subseteq \mathbb{P}^n$, $W \subseteq \mathbb{P}^m$ irreducible varieties. Then, V & W are birationally isomorphic if, and only if, $\mathbb{K}(V)$ & $\mathbb{K}(W)$ are isomorphic as fields.

Proof: We use coordinates $[x_0:\cdots:x_n]$ for $\mathbb{P}^n$ & $[y_0:\cdots:y_m]$ for $\mathbb{P}^m$.

Fix k, l with $V \cap U_k \neq \emptyset$ & $W \cap U_\ell \neq \emptyset$. By Corollary 2 §20.1, we have

$$\mathbb{K}(V) \simeq \mathbb{K}(V \cap U_k) = \mathbb{K}\left(\frac{x_0}{x_k}, \ldots, \frac{x_n}{x_k}\right) \quad \text{as fields.}$$

$$\mathbb{K}(W) \simeq \mathbb{K}(W \cap U_\ell) = \mathbb{K}\left(\frac{y_0}{y_\ell}, \ldots, \frac{y_m}{y_\ell}\right)$$

By Corollary 3 §11.2 the affine varieties $V \cap U_k \subseteq U_k \simeq \mathbb{A}^n$ & $W \cap U_\ell \subseteq U_\ell \simeq \mathbb{A}^m$

are birational $\Leftrightarrow$ $\mathbb{K}(V\cap U_k) \simeq \mathbb{K}(W\cap U_\ell)$ as fields.

($\varphi: V\cap U_k \dashrightarrow W\cap U_\ell$ $\Leftrightarrow$ $\varphi^*: \mathbb{K}(W\cap U_\ell) \longrightarrow \mathbb{K}(V\cap U_k)$)

$$\frac{f}{g} \longmapsto \frac{f\circ\varphi}{g\circ\varphi}$$

<u>Claim</u>: The same map $\varphi: V\cap U_k \dashrightarrow W\cap U_\ell$ gives a birational map $\varphi: V \dashrightarrow W$ since $V\cap U_k \subseteq V$ is a dense open set in V & $W\cap U_\ell \subseteq W$ is a dense open set in W

Pf/ The isomorphism $\varphi^* \, \mathbb{K}(W\cap U_\ell) \xrightarrow{\sim} \mathbb{K}(V\cap U_k)$ expresses $\frac{\bar{y}_j}{\bar{y}_\ell}$ as a rational function $\frac{f_j}{g_\ell}\left(\frac{\bar{x}_0}{\bar{x}_k}, \ldots, \frac{\bar{x}_n}{\bar{x}_k}\right)$, so $\varphi = \left(\frac{f_0}{g_\ell}, \ldots, \frac{f_n}{g_\ell}\right)$ is our map on $V\cap U_k$.

$g_\ell\left(\frac{\bar{x}_0}{\bar{x}_k}, \ldots, \frac{\bar{x}_n}{\bar{x}_k}\right)$ is not identically 0 on $\mathbb{K}[V\cap U_k]$, meaning $g_\ell \notin I(V\cap U_k)$

To lift this to a rational map we homogenize f_j & g_ℓ with respect to x_k. This gives

$$\frac{\bar{y}_j}{\bar{y}_\ell} = \frac{\bar{x}_k^{d'} F_j(\bar{x}_0, \ldots, \bar{x}_n)}{\bar{x}_k^{d_j} G_\ell(\bar{x}_0, \ldots, \bar{x}_n)} \qquad d_j = \deg f_j \;\;\&\;\; d' = \deg g_\ell.$$

Since f_j & g_ℓ are degree 0 rational functions in $\bar{x}_0 \ldots \bar{x}_n$.

Setting $A_\ell = x_k^{d_j} G_\ell$ & $A_j = x_k^{d'} F_j$ for $j \neq \ell$ induces a rational map $\varphi: V \dashrightarrow \mathbb{P}^m$ $\varphi = [A_0 : \cdots : A_m]$.

This map is defined on a dense open set of $V\cap U_k$, which in turn is dense in V

In particular $\overline{\operatorname{im}(\varphi)} = \overline{\operatorname{im}(\varphi|_{V\cap U_k})} = \overline{W\cap U_\ell} = W$, so $\varphi: V \dashrightarrow W$. □

§1 The case of $\overline{\mathbb{K}} = \mathbb{K}$:

Exploiting the characterization of $\mathcal{O}_X$ for $X \subseteq \mathbb{A}^n_{\mathbb{K}}$ irreducible affine variety when $\overline{\mathbb{K}} = \mathbb{K}$ (see Theorem 1 §13.1) we get:

Theorem 2: Fix $V \subseteq \mathbb{P}^n$ irreducible & $\bar{G} \in S(V)_d \smallsetminus \{0\}$. Then, $\mathcal{O}_V(D(G))$ is the 0^{th} graded piece of the localization $S(V)[G^{-1}]$. In particular, if $p \in V$ then

$$\mathcal{O}_{V,p} \simeq \mathcal{O}_{V\cap U_k, p} \simeq \mathbb{K}[V\cap U_k]_{\tilde{m}_p}$$

for any $k = 0, \ldots n$ with $p \in U_k$. Here, $\tilde{m}_p \subseteq \mathbb{K}[V\cap U_k]$ is the maximal ideal of those regular functions on $V\cap U_k$ vanishing on p. (<u>Note</u>: $V\cap U_k \subseteq U_k \simeq \mathbb{A}^n_{\mathbb{K}}$ is irreducible & non-empty)

• In particular, we can characterize regular functions on $\mathbb{P}^n$ if $\overline{\mathbb{K}}=\mathbb{K}$

Corollary 1: If $\overline{\mathbb{K}}=\mathbb{K}$, we have $\mathcal{O}_{\mathbb{P}^n}(\mathbb{P}^n) = \mathbb{K}$

Proof: Take $G=1$ so $S(\mathbb{P}^n)_{[1^{-1}]} = S(\mathbb{P}^n) = \mathbb{K}[x_0,..,x_n]$ with the standard grading. The 0^{th} graded piece of $\mathbb{K}[x_0,..,x_n]$ is $\mathbb{K}$.

Example: On $\mathbb{P}^1$: $\mathcal{O}_{\mathbb{P}^1}(U_0) = \mathbb{K}[z]$ & $\mathcal{O}_{\mathbb{P}^1}(U_1) = \mathbb{K}[z^{-1}]$.
The only polynomial functions in both z & z^{-1} are the constant ones.

• Similar considerations follow for rational maps.

§2. The Veronese map:

Definition: Given $d,n \geq 1$ we define the Veronese map of degree d on $\mathbb{P}^n$ as

$$\nu_d : \mathbb{P}^n \longrightarrow \mathbb{P}^{\binom{n+d}{d}-1}$$
$$[\underline{x}] \longrightarrow [\underline{x}^I]_I$$

where $x^I = x_{i_0} \cdots x_{i_d}$ for $I = \{i_0,..,i_d\} \subseteq \{0,..,n\}$ (I multiset) we write $\left(\binom{[n]}{d}\right)$ for the number of multisets of $\{0,...n\}$ of size d. This number is $\binom{n+d}{d}$.

The coordinates of $\nu_d([\underline{x}])$ correspond to all monomials of degree d in $n+1$ variables.

Remark: Degree d hypersurfaces in $\mathbb{P}^n$ are $V_{proj}(F)$ for F of degree d. They correspond to hyperplane sections of $\nu_d(\mathbb{P}^n)$

Special case: $n=1$:

$$\nu_d : \mathbb{P}^1 \longrightarrow \mathbb{P}^{\binom{d+1}{d}-1} = \mathbb{P}^d$$
$$[s:t] \longmapsto [s^d : s^{d-1}t : \ldots : t^d]$$

image is called the rational normal curve in $\mathbb{P}^d$.

We write $[u_0:\ldots:u_d]$ for a point in $\mathbb{P}^d$

Last time, we discussed the case $d=2$ $n=1$ at length. The statement for $d=2$ translates to analogous statements for $d>2$. More precisely,

(1) ν_d is injective ($\frac{u_1}{u_0}$ gives $\frac{x_1}{x_0}$, $\frac{u_d}{u_{d-1}}$ gives $\frac{x_0}{x_1}$ & one of these is nonzero)

(2) $V = V_{proj}(\{u_i u_j - u_k u_\ell : i+j=k+\ell\}) \subseteq \mathbb{P}^d$ satisfies $\nu_d(\mathbb{P}^1) = V$

- $(\subseteq)$ is clear ($u_i = s^{d-i} t^i \quad \forall i$)
- For $(\supseteq)$ we need the following results:

Lemma 1: $V \subseteq U_0 \cup U_d$. ($U_i = \{u_i \neq 0\} \subseteq \mathbb{P}^d$ is a standard affine patch)

Proof: By contradiction, assume $\exists\, [a] \in V \setminus (U_0 \cup U_d)$, so $a_0 = a_d = 0$.

- The relations $u_0 u_j - u_k u_\ell = 0$ applied to $k = \ell \leq \lfloor \frac{d}{2} \rfloor$ & $j = 2k$ give $u_k = 0$ for all $0 \leq k \leq \lfloor \frac{d}{2} \rfloor$.

  The relations $u_d u_j - u_k u_\ell = 0$ for $k = \ell \geq \lfloor \frac{d}{2} \rfloor$ & $j = 2k - d$ give $u_k = 0$ for all $\lfloor \frac{d}{2} \rfloor \leq k \leq d$.

Conclude: $\underline{u} = 0 \in \mathbb{P}^n$ Contradiction! □

Lemma 2: There exists a regular map $\tau : V \longrightarrow \mathbb{P}^1$ with $\tau = \nu_d^{-1}$.

Proof: Since $V \subseteq U_0 \cup U_d$, we need only define the restrictions $\tau_0 = \tau|_{U_0}$ & $\tau_d = \tau|_{U_d}$ & check they agree on $U_0 \cap U_d$.

- For $\tau_d^{(0)} : V \cap U_0 \longrightarrow \mathbb{P}^1$ $\quad \tau_0[\underline{u}] = [u_0 : u_1]$
- For $\tau_d^{(1)} : V \cap U_d \longrightarrow \mathbb{P}^1$ $\quad \tau_d[\underline{u}] = [u_{d-1} : u_d]$

On $U_0 \cap U_d$, both $u_0, u_d \neq 0$ & the relation $u_0 u_d - u_1 u_{d-1} = 0$ ensures $[u_0 : u_1] = [u_{d-1} : u_d]$ in $\mathbb{P}^1$.

Conclude: τ is a regular map on V

Furthermore, $$\tau \circ \nu_d([s:t]) = \begin{cases} [s^d : s^{d-1} t] = [s:t] & \text{on } s \neq 0 \\ [s t^{d-1} : t^d] = [s:t] & \text{on } t \neq 0 \end{cases}$$

Next, we need to check $\nu_d \circ \tau = id_V$ so $V \subseteq \nu_d(\mathbb{P}^1)$.

- **On U_0:** $\nu_d \circ \tau_d^{(0)}([\underline{u}]) = \nu_d([u_0 : u_1]) = [u_0^d : u_0^{d-1} u_1 : u_0^{d-2} u_1^2 : \cdots : u_1^d]$

Claim: On $V \cap U_0$: $[u_0^d : u_0^{d-1} u_1 : u_0^{d-2} u_1^2 : \cdots : u_1^d] = \underset{\underset{\mathbb{K}\setminus\{0\}}{\cap}}{u_0^{d-1}} [\underline{u}]$.

Pf/ • $u_0 u_2 = u_1^2$ so $u_0^{d-1} u_2 = u_0^{d-2} u_1^2$

• $u_0 u_3 = u_1 u_2 = u_1 \frac{u_1^2}{u_0} = \frac{u_1^3}{u_0}$ so $u_0^2 u_3 = u_1^3 \rightsquigarrow u_0^{d-1} u_3 = u_0^{d-3} u_1^3$

Inductively on k: $\boxed{u_0^{d-1} u_k = u_0^{d-k} u_1^k}$

$\left(u_0 u_k = u_1 u_{k-1} \underset{IH}{=} u_1 \frac{(u_0^{d-(k-1)} u_1^{k-1})}{u_0^{d-1}} = u_1^k u_0^{2-k} \text{ so } u_0^{d-1} u_k = u_0^{d-k} u_1^k \right)$

- **On U_1:** $\nu_d \circ \tau_d^{(1)}([\underline{u}]) = \nu_d([u_{d-1} : u_d]) = [u_{d-1}^d : u_{d-1}^{d-1} u_d : u_{d-1}^{d-2} u_d^2 : \cdots : u_d^d]$

Claim: On $V \cap U_1$: $[u_{d-1}^d : u_{d-1}^{d-1} u_d : u_{d-1}^{d-2} u_d^2 : \cdots : u_d^d] = \underset{\underset{\mathbb{K}\setminus\{0\}}{\cap}}{u_d^{d-1}} [\underline{u}]$.

The argument is symmetric to the one we used starting from the last coordinate & using $u_d u_{d-j-1} = u_{d-1} u_{d-j}$ $\forall j$ to conclude $u_d^{d-1} u_{d-j} = u_d^{d-j} u_{d-1}^j$

Corollary 2: $\nu_d(\mathbb{P}^1) \simeq \mathbb{P}^1$ hence the name rational.

• The results for $\nu_d : \mathbb{P}^n \longrightarrow \mathbb{P}^{\binom{n+d}{d}-1}$ are analogous as those for $n=1$

Theorem 3: (1) The map ν_d is injective.

(2) The image $\nu_d(\mathbb{P}^n)$ is a projective variety in $\mathbb{P}^{\binom{n+d}{d}-1}$. Furthermore,

$$\nu_d(\mathbb{P}^n) = V_{proj}(\{x^I x^J - x^K x^L : I \sqcup J = K \sqcup L\}) =: V_{d,n} \quad (*)$$

(to allow repetitions)

We call it the Veronese variety.

(3) ν_d is an isomorphism.

Proof The arguments are essentially those from the $n=1$ case. The only difficulty is in doing the bookkeeping.

(1) Set $N = \binom{n+d}{d} - 1$

- First $\nu_d([\underline{x}]) = \nu_d([\underline{y}])$ in $\mathbb{P}^N$ forces $[\underline{x}]$ & $[\underline{y}]$ to lie on the same standard affine patches (Since $\exists\, \lambda \in \mathbb{K} \setminus \{0\}$ with $x_k^d = \lambda y_k^d \ \forall k$)

without loss of generality, assume $[x], [y] \in U_0$ & set $x_0 = y_0 = 1$.

- Since $\nu_d([x]) = \nu_d([y])$ in $\mathbb{P}^n$ $\exists\, \lambda \in \mathbb{K}^*$ with $\nu_d([x])_I = \lambda\, \nu_d([y])_I$

  $\forall$ size d multiset I in $\{0, \ldots, n\}$.

In particular, $I = \underbrace{\{0, \ldots, 0\}}_{d \text{ times}}$ gives $1 = x_0^d = \lambda y_0^d = \lambda$ so $\lambda = 1$

In turn, $I = \{\underbrace{0, 0, \ldots 0}_{d-1}, \ell\}$ gives $x_\ell = x_0^{d-1} x_\ell = \lambda y_0^{d-1} y_\ell = 1 \cdot 1^{d-1} y_\ell = y_\ell$

<u>Conclude</u>: $[\underline{x}] = [1 : x_1 : \cdots : x_n] = [1 : y_1 : \cdots : y_\ell] = [\underline{y}]$. so ν_d is injective.

(2) & (3): We note $\nu_d(\mathbb{P}^n) \subseteq$ (RHS) of (*) by construction. For the converse, we follow the same strategy used for $n = 1$

<u>STEP 1</u>: Show $V_{d,n} \subseteq U_{0^d} \cup \cdots \cup U_{n^d}$

Pf/ Argue by contradiction + induction on $n+d$

Pick $[u] \in V \setminus \bigcup_{k=0}^{n} U_{k^d}$. We break the coordinates of $\mathbb{P}^{\binom{n+d}{d}-1}$

into $\mathcal{J}_j = \{$multisets containing j copies of $n\}$

<u>Claim</u>: $u_I = 0 \quad \forall\, I \in \mathcal{J}_j \quad \forall j = 0, \ldots, d-1$

Indeed, consider the projection

$$\pi_j : \mathbb{P}^{\binom{n+d}{d}-1} \dashrightarrow \mathbb{P}^{\binom{n-1+(d-j)}{d-j}-1}$$
$$[x_I]_I \longmapsto [x_{I \setminus j[n]}]_{I \in \mathcal{J}_j}$$

- The map is <u>not</u> defined if $x_{I \setminus j[n]} = 0 \ \forall\, I \in \mathcal{J}_j$, so it is rational
- By construction: $\pi_j(V_{d,n}) \subseteq V_{d-j, n-1}$.

If $u \in$ Domain of π_j, then $\pi_j(\underline{u}) \in \underbrace{V_{d-j, n-1} \setminus \bigcup_{k=0}^{n-1} U_{k^d}}_{= \emptyset \text{ by (IH)}}$ <u>Contd!</u>

So $u_I = 0 \quad \forall I \in \mathcal{Y}_j$, as we wanted.

Conclude: $u_I = 0 \quad \forall I \in \binom{[n]+d}{d}$ ($u_{n^d} = 0$ by assumption & the rest follow by the claim)

STEP 2: Define $\tau_d : V_{d,n} \longrightarrow \mathbb{P}^n$ regular & show $\tau_d = \nu_d^{-1}$.

Pf/ We define τ_d on each affine patch U_{k^d} & induct on $n+d$ to show the validity of the statement.

$$\tau_d^{(k)} : \tau_d|_{U_{k^d}} : U_{k^d} \longrightarrow \mathbb{P}^n$$
$$[\underline{u}] \longmapsto [u_{0k^{d-1}} : u_{1k^{d-1}} : \dots : u_{k^d} : \dots : u_{k^{d-1}n}]$$

- We need to show $\underline{\tau_d \text{ is a regular map}}$.

For this we pick $k \neq \ell$ & show $\tau_d^{(k)}|_{U_{k^d} \cap U_{\ell^d}} = \tau_d^{(\ell)}|_{U_{\ell^d} \cap U_{k^d}}$

(with $U_{k^d} \cap U_{\ell^d} \subseteq U_{k^d}$ and $U_{\ell^d} \cap U_{k^d} \subseteq U_{\ell^d}$)

Using $\begin{cases} u_{\ell k^{d-1}} \, u_{\ell^{d-1} k} = u_{\ell^d} \, u_{k^d} \\ u_{\ell^d} \, u_{k^{d-1} s} = u_{\ell^{d-1} s} \, u_{k^{d-1} \ell} \quad \forall s. \end{cases}$ we get that

$$\tau_d^{(k)}|_{U_{k^d} \cap U_{\ell^d}} [u] = \tau_d^{(\ell)}|_{U_{\ell^d} \cap U_{k^d}} [u] \text{ for all } [\underline{u}] \in U_{k^d} \cap U_{\ell^d} = U_{\ell^d} \cap U_{k^d}$$

Indeed:

$$[u_{0k^{d-1}} : \dots : u_{k^d} : u_{k^{d-1}n}] = \left[u_{0k^{d-1}} : \dots : \frac{u_{\ell k^{d-1}} u_{\ell^{d-1}k}}{u_{\ell^d}} : \dots : u_{k^{d-1}n}\right]$$

(where $\frac{u_{\ell k^{d-1}} u_{\ell^{d-1}k}}{u_{\ell^d}} =: u_{k^d}$)

$$= [u_{\ell^d} u_{0k^{d-1}} : u_{\ell^d} u_{1k^{d-1}} : \dots : u_{\ell k^{d-1}} u_{\ell^{d-1}k} : \dots : u_{\ell^d} u_{k^{d-1}n}]$$

$$= [u_{\ell^{d-1}0} \boxed{u_{k^{d-1}\ell}} : u_{\ell^{d-1}1} \boxed{u_{k^{d-1}\ell}} : \dots : u_{\ell^{d-1}k} \boxed{u_{k^{d-1}\ell}} : \dots : u_{\ell^{d-1}n} \boxed{u_{k^{d-1}\ell}}]$$

$(s=0) \quad (s=1) \quad \dots \quad (s=n)$

$$= [u_{\ell^{d-1}0} : \dots : u_{\ell^{d-1}n}] \text{ since } \boxed{u_{k^{d-1}\ell}} \neq 0 \text{ because } u_{\ell^d} u_{k^d} = u_{\ell^{d-1}k} u_{k^{d-1}\ell}$$

($u_{\ell^d} u_{k^d} \neq 0$ on $U_{k^d} \cap U_{\ell^d}$)

- To finish we show that $\tau_d = \nu_d^{-1}$. This will imply $\nu_d(\mathbb{P}^n) = V_{d,n}$.

- **Claim 1:** $\tau_d \circ \nu_d = id_{\mathbb{P}^n}$.

Pf/ We check it on each affine patch of $\mathbb{P}^n$. **Note:** $\nu_d(U_k) \subseteq U_{k^d}$

So $\tau_d \circ \nu_d|_{U_k}[x] = \tau_d|_{U_{k^d}} \circ \nu_d|_{U_k}[x] = [u_{0k^{d-1}} : \cdots : u_{k^{d-1}n}]$

where $u_{jk^{d-1}} = (\nu_d[x])_{jk^{d-1}} = x_j\, x_k^{d-1}$ $\forall j$.

So $[u_{0k^{d-1}} : \cdots : u_{k^{d-1}n}] = [x_0 \boxed{x_k^{d-1}} : \cdots : x_n \boxed{x_k^{d-1}}] = [x_0 : \cdots : x_n]$

because $\boxed{x_k^{d-1}} \neq 0$ on $U_k \subseteq \mathbb{P}^n$.

- **Claim 2:** $\nu_d \circ \tau_d|_{U_{k^d}}([u]) = [u]$

Pf/ By construction, $\tau_d([u]) \in U_k = \{x_k \neq 0\} \subseteq \mathbb{P}^n$ since $x_k = u_{k^d}$.

$\nu_d \circ \tau_d^{(k)}([u]) = [x^I]_I$ where $x_i = u_{ik^{d-1}}$. We need to show $[x^I]_I = [u_I]_I$ in $\mathbb{P}^{\binom{n+d}{d}-1}$, ie $\exists \lambda \in \mathbb{K}^*$ with $x^I = \lambda u_I$ $\forall I \in \binom{[n]+d}{d}$.

In particular, for $I = k^d$ we get $x^{k^d} = (x_k)^d = (u_{k^d})^d = \lambda u_{k^d} \iff \lambda = u_{k^d}^{d-1}$.

We claim: $\boxed{x_I = (u_{k^d})^{d-1} u_I \quad \forall I}$ $(*)$

To prove this statement we partition $\binom{[n]+d}{d}$ by the number of times k appears in the multiset. More precisely, for each $j = 0, \ldots, d$ we define

$$K_j := \{ I \in \binom{[n]+d}{d} : k^j \subseteq I \text{ \& } k^{j+1} \not\subseteq I \}$$

By construction $\bigcup_{j=0}^{d} K_j = \binom{[n]+d}{d}$ & $K_d = \{k^d\}$.

We prove the statement $(*)$ on each K_j by reverse induction on $j \in \{0, \ldots, d\}$ using the relations in $V_{d,n}$.

- **Base case:** $j = d$ Follows from $x^{k^d} = (u_{k^d})^d$.

- **Inductive Step:** Fix $0 \leq j < d$ & $I \in K_j$. Write $I = I' \cup \{k^j\}$

For $i_0 \in I'$ set $I'' = I' \setminus \{i_0\}$

Write $J := k^d \in K_d$, $A := i_0 k^{d-1} \in K_{d-1}$ & $B := I'' \cup k^{j+1} \in K_{j+1}$

Since $A \cup B = I \cup J$, the relations on $V_{d,n}$ imply $u_I u_J = u_A u_B$.

Equivalently $u_I = u_A u_B / u_J$ since $u_J = u_{k^d} \neq 0$ on U_{k^d}. (**)

We check the claim on x^I using this identity and the definition of ν_d.

$$x^I = x^{I'} x^{k^j} = x_{i_0} x^{I''} x_k^j \underset{\substack{\downarrow \\ \text{multiply \& divide by } x_k^d = u_{k^d}^d \neq 0}}{=} u_{i_0 k^{d-1}} \frac{x^{I''} x_k^j \, x_k^d}{x_k^d} \underset{\substack{\downarrow \\ d \geq 1}}{=} \frac{u_{i_0 k^{d-1}} \, x^{I''} x_k^{j+1} \, x_k^{d-1}}{x_k^d}$$

$$= \frac{u_{i_0 k^{d-1}} \, x^{I'' k^{j+1}}}{x_k} = \frac{u_A \, x^B}{u_{k^d}} \underset{\substack{\downarrow \\ \text{(IH) on } B \in K_{j+1}}}{=} \frac{u_A \, u_{k^d}^{d-1} \, u_B}{u_J} = \boxed{\frac{u_A u_B}{u_J}} \, u_{k^d}^{d-1} = u_{k^d}^{d-1} u_I$$

$= u_I$ by (**) ∎