Lecture XXIII: Products of Varieties & the Segre embedding

Recall: In our wishlist for properties of affine varieties, we wanted to have fiber products. For this to happen, we need products of varieties to be varieties, both in the affine and the projective case. Later on, we'll discuss the situation for abstract varieties (obtained by gluing affine varieties)

§1 Products of Affine varieties:

Recall: $\mathbb{A}^m \times \mathbb{A}^n$ can be viewed as the affine variety $\mathbb{A}^{n+m}$ via the map:

$$\begin{array}{ccc} \mathbb{A}^m \times \mathbb{A}^n & \longrightarrow & \mathbb{A}^{n+m} \\ (\underline{x}\ ,\ \underline{y}) & \longmapsto & (\underline{x}, \underline{y}) \end{array}$$

Similarly, if $V \subseteq \mathbb{A}^m$, $W \subseteq \mathbb{A}^n$ are affine varieties, we can view the set $V \times W$ as an affine variety in $\mathbb{A}^{n+m}$. Indeed

Proposition 1: $V \times W = V(I) \subseteq \mathbb{A}^{m+n}$ where $I = I(V)\,\mathbb{K}[x_1, \ldots x_{n+m}] + I(W)\,\mathbb{K}[x_1, \ldots x_{n+m}]$

Proof: $(\underline{x}, \underline{y}) \in V \times W \iff \underline{x} \in V$ & $\underline{y} \in W \iff \begin{array}{ll} f(\underline{x}) = 0 & \forall f \in I(V) \\ g(\underline{y}) = 0 & \forall g \in I(W) \end{array}$

$$\underset{\text{def of } I}{\Longrightarrow} \quad h(\underline{x}, \underline{y}) = 0 \quad \forall\ h \in I$$

For the converse, we show $V(I) \subseteq V \times W$

Since $I(V)\,\mathbb{K}[x_1, \ldots x_{n+m}] \subseteq I$, then $V(I) \subseteq V(I(V)\,\mathbb{K}[x_1, \ldots x_{n+m}]) = V \times \mathbb{A}^n$

$I(W)\,\mathbb{K}[x_1, \ldots, x_{n+m}] \subseteq I$ then $V(I) \subseteq V(I(W)\,\mathbb{K}[x_1, \ldots x_{n+m}]) = \mathbb{A}^m \times W$

Thus $V(I) \subseteq (V \times \mathbb{A}^n) \cap (\mathbb{A}^m \times W) = V \times W$.

§2. Segre embeddings:

⚠ The map $\mathbb{A}^m \times \mathbb{A}^n \longrightarrow \mathbb{A}^{n+m}$ does <u>not</u> extend to a map $\mathbb{P}^{m-1} \times \mathbb{P}^{n-1} \longrightarrow \mathbb{P}^{m+n-2}$ since the scalars used in the definition of $\mathbb{P}^{n-1}$ & $\mathbb{P}^{m-1}$ are separated

$$(x,y) \sim (\lambda x, \mu y) \longmapsto (\lambda x, \mu y) \not\sim [x;y] \qquad \text{for } \lambda, \mu \in \mathbb{K}^*.$$

- A map $\mathbb{P}^n \times \mathbb{P}^m \longrightarrow \mathbb{P}^N$ will need interaction between the coordinates of $\mathbb{P}^n$ & $\mathbb{P}^m$. This is precisely what Segre embeddings do:

Definition: The Segre embedding is the set map

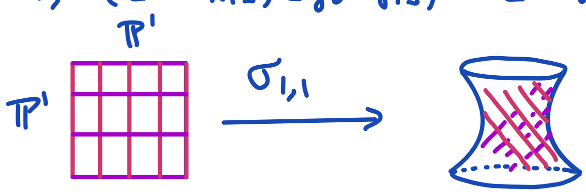
$$\sigma_{m,n} : \mathbb{P}^m \times \mathbb{P}^n \longrightarrow \mathbb{P}^{(n+1)(m+1)-1}$$
$$([x],[y]) \longmapsto [x_i y_j]_{i,j}$$

We label the coordinates of $\mathbb{P}^{(n+1)(m+1)-1}$ as $(z_{ij})_{i,j}$

Lemma 1: $\sigma_{m,n}$ is well-defined

Pf/ $\sigma_{m,n}(([\lambda \underline{x}],[\mu y])) = [\lambda\mu\, x_i y_j]_{i,j} \overset{\lambda\mu \neq 0}{=} [x_i y_j] = \sigma_{m,n}([x],[y])$ for each $\lambda, \mu \in \mathbb{K}^*$.

Example: $n=m=1$ $\qquad \sigma_{1,1}([x_0:x_1],[y_0:y_1]) = [x_0y_0 : x_0y_1 : x_1y_0 : x_1y_1] \in \mathbb{P}^3$



= Segre variety in $\mathbb{P}^3$.

Note: • A line $\{a\} \times \mathbb{P}^1$ maps to a line $[a_0y_0 : a_0y_1 : a_1y_0 : a_1y_1]$ in $\mathbb{P}^3$
$$= y_0 \underbrace{[a_0 : 0 : a_1 : 0]}_{=v_1} + y_1 \underbrace{[0 : a_0 : 0 : a_1]}_{=v_2}$$

where $[y_0:y_1] \in \mathbb{P}^1$. By construction, $\{v_1, v_2\}$ are linearly independent over $\mathbb{K}$.

• Similarly, a line $\mathbb{P}^1 \times \{b\}$ maps to the line $x_0 \underbrace{[b_0 : b_1 : 0 ; 0]}_{=w_1} + x_1 \underbrace{[0:0:b_0:b_1]}_{=w_2}$ in $\mathbb{P}^3$, where $[x_0:x_1] \in \mathbb{P}^1$ & $\{w_1, w_2\}$ are l.i over $\mathbb{K}$.

Remark: image $\sigma_{1,1} \subseteq V(z_{00}z_{11} - z_{01}z_{10}) := V$. In fact, equality holds because we can build $\tau_{1,1} = \sigma_{1,1}^{-1} : V \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$

In addition, the 2 families of lines constructed above lie in V. They are known as the rulings of the quadric hypersurface V

• Next, we build the map $\tau_{1,1}$ in each chart of $\mathbb{P}^3$:

• On $U_{ij} = \{z_{ij} \neq 0\} \subseteq \mathbb{P}^3$ $i,j = 0$ or 1, we define $\tau_{1,1}^{(i,j)} = \tau_{1,1}|_{U_{ij}}$ exploiting our underlying assumption that $z_{ij} = x_i y_j$ on $\mathrm{im}(\sigma_{1,1})$. Thus we will be able to extract a factor y_j from the first entry pair of $\tau_{1,1}^{(i,j)}$ & a factor x_i from the second entry pair.

More precisely,
$$\tau_{1,1}^{(0,0)}([\underline{z}]) = ([z_{00}:z_{10}], [z_{00}:z_{01}])$$
$$\tau_{1,1}^{(0,1)}([\underline{z}]) = ([z_{01}:z_{11}], [z_{00}:z_{01}])$$
$$\tau_{1,1}^{(1,0)}([\underline{z}]) = ([z_{00}:z_{10}], [z_{10}:z_{11}])$$
$$\tau_{1,1}^{(1,1)}([\underline{z}]) = ([z_{01}:z_{11}], [z_{10}:z_{11}])$$

• <u>Claim 1:</u> $\sigma_{1,1} \circ \tau_{1,1} = id_V$ $(\Rightarrow V \subseteq \mathrm{im}(\sigma_{1,1}))$

Pf/ We show $\sigma_{1,1} \circ \tau_{1,1}|_{U_{ij}} = id_{V \cap U_{ij}}$ by working in each chart U_{ij} of $\mathbb{P}^3$.

For example, $\sigma_{1,1} \circ \tau_{1,1}^{(0,0)} = id_{U_{00} \cap V}$ since $z_{01}z_{10} = z_{00}z_{11}$ in $U_{00} \cap V$

$$\sigma_{1,1} \circ \tau_{1,1}^{(0,0)}([\underline{z}]) = \sigma_{1,1}([z_{00}:z_{10}], [z_{00}:z_{01}]) = [z_{00}^2 : z_{00}z_{01} : z_{10}z_{00} : z_{01}z_{10}]$$
$$= [z_{00}^2 : z_{00}z_{01} : z_{00}z_{10} : z_{00}z_{11}] \underset{z_{00} \neq 0}{=} [\underline{z}].$$

On the other patches, the computation is similar, we get $[z_{ij}\,\underline{z}] = [\underline{z}]$ since $z_{ij} \neq 0$ on U_{ij}.

• <u>Claim 2:</u> $\tau_{1,1} \circ \sigma_{1,1} = id_{\mathbb{P}^1 \times \mathbb{P}^1}$.

Pf/ We check that this is true on each set $U_i \times U_j \subseteq \mathbb{P}^1 \times \mathbb{P}^1$! Since the sets cover $\mathbb{P}^1 \times \mathbb{P}^1$, this is enough. Note that $\sigma_{1,1}(U_i \times U_j) \subseteq U_{ij}$ so we only need to show $\tau_{1,1}^{(i,j)} \circ \sigma_{1,1}|_{U_{ij}} = id_{U_i \times U_j}$. We do this calculation in each of the 4 cases.

For example, $$\tau_{1,1}^{(0,0)} \circ \sigma_{1,1}|_{U_0 \times U_0}([\underline{x}],[\underline{y}]) = \tau_{1,1}^{(0,0)}([x_0y_0 : x_0y_1 : x_1y_0 : x_1y_1])$$
$$= ([x_0y_0 : x_1y_0], [x_0y_0 : x_0y_1])$$
$$\underset{x_0, y_0 \neq 0}{=} ([x_0:x_1], [y_0:y_1])$$

On the other charts we get a similar situation; we will be off by a factor of y_j in the first entry & by x_i in the second entry. These scalars are nonzero, so they can be removed.

$$\tau_{1,1}^{(i,j)} \circ \sigma_{1,1}|_{U_i \times U_j}([x],[y]) = ([x_0 y_j : x_1 y_j], [x_i y_0 : x_i y_1]) = ([\underline{x}],[\underline{y}]) \quad \square$$

$\hookrightarrow x_i, y_j \neq 0$

• The statements for general m & n are similar:

Theorem 1: The map $\sigma_{n,m}$ gives a bijection between $\mathbb{P}^m \times \mathbb{P}^n$ and the projective variety $V = V(I) \subseteq \mathbb{P}^{mn+m+n}$, where $I \subseteq \mathbb{K}[z_{ij} : \begin{smallmatrix} i=0,\dots,m \\ j=0,\dots,n \end{smallmatrix}]$ is the (homogeneous) ideal $I = \langle z_{ij} z_{k\ell} - z_{i\ell} z_{kj}$ for $\begin{smallmatrix} i\neq k \\ j \neq \ell \end{smallmatrix}$ $\begin{smallmatrix} i,k \in \{0,\dots,m\} \\ j,\ell \in \{0,\dots,n\} \end{smallmatrix}\rangle$

Moreover V is irreducible.

Proof: We build an inverse map $\tau_{m,n} : V \longrightarrow \mathbb{P}^m \times \mathbb{P}^n$ to $\sigma_{m,n}$.

We define it on each coordinate patch $U_{k\ell}$ of $\mathbb{P}^{mn+m+n}$.

$$\tau_{m,n}^{(k,\ell)} = \tau_{m,n}|_{V \cap U_{k,\ell}} : V \cap U_{k,\ell} \longrightarrow \mathbb{P}^m \times \mathbb{P}^n.$$
$$\underline{z} \longmapsto ([z_{i\ell}]_{i=0}^m ; [z_{kj}]_{j=0}^n)$$

• Claim 1: The map is well-defined, ie, it agrees on the overlaps.

Pf/ To prove this, we analyze 3 cases:

(1) $\tau_{m,n}^{(k,\ell)} = \tau_{m,n}^{(k,q)}$ on $V \cap U_{k\ell} \cap U_{kq}$ for $q \neq \ell$.

We need only check the 1st coordinate ie we need to check $[z_{i\ell}]_{i=0}^m = [z_{iq}]_{i=0}^m$ in $\mathbb{P}^m$ for all $z \in V \cap U_{k\ell} \cap U_{kq}$.

The relations on V ensure that $z_{i\ell} z_{kq} - z_{iq} z_{k\ell} = 0$

Since $z_{k\ell}, z_{kq} \neq 0$ on $V \cap U_{k\ell} \cap U_{kq}$ we get $z_{i\ell} = \boxed{\dfrac{z_{k\ell}}{z_{kq}}} z_{iq} \quad \forall i = 0,\dots,m$

Thus $[z_{i\ell}]_i = [z_{iq}]_i$ in $\mathbb{P}^m$.

(2) $\tau_{m,n}^{(k,\ell)} = \tau_{m,n}^{(p,\ell)}$ on $V \cap U_{k\ell} \cap U_{p\ell}$ for $p \neq k$.

We need only check the 2nd coordinate ie we need to check $[z_{kj}]_{j=0}^n = [z_{pj}]_{j=0}^n$ in $\mathbb{P}^n$

for all $z \in V \cap U_{k\ell} \cap U_{p\ell}$.

The relations on V ensure that $z_{pj} z_{k\ell} - z_{p\ell} z_{kj} = 0$

Since $z_{k\ell}, z_{p\ell} \neq 0$ on $V \cap U_{k\ell} \cap U_{p\ell}$ we get $z_{kj} = \boxed{\frac{z_{k\ell}}{z_{p\ell}}} z_{pj}$ $\forall i = 0, \dots, m$

Thus $[z_{kj}]_j = [z_{pj}]_j$ in $\mathbb{P}^m$.

(3) $\tau^{(k,\ell)}_{m,n} = \tau^{(p,q)}_{m,n}$ on $V \cap U_{k\ell} \cap U_{pq}$ for $k \neq p$ & $\ell \neq q$

We check both coordinates. Note that $\underbrace{z_{k\ell} z_{pq}}_{\neq 0} - z_{kq} z_{p\ell} = 0$ on V forces $z_{kq}, z_{p\ell} \neq 0$. on $V \cap U_{k\ell} \cap U_{pq}$.

1st coord : $[z_{i\ell}]_i \overset{?}{=} [z_{iq}]_i$ in $\mathbb{P}^m$. Use $z_{i\ell} z_{pq} - z_{iq} z_{p\ell} = 0$

to conclude that $z_{i\ell} = \underbrace{\boxed{\frac{z_{p\ell}}{z_{pq}}}}_{\in \mathbb{K}^*} z_{iq}$ $\forall i = 0, \dots, m$, ie $[z_{i\ell}]_i = [z_{iq}]_i$ in $\mathbb{P}^m$.

2nd coord : $[z_{kj}]_j \overset{?}{=} [z_{pj}]$ in $\mathbb{P}^n$. Use $z_{kj} z_{pq} - z_{kq} z_{pj} = 0$

to conclude that $z_{kj} = \underbrace{\boxed{\frac{z_{kq}}{z_{pq}}}}_{\in \mathbb{K}^*} z_{pj}$ $\forall j = 0, \dots, n$, ie $[z_{kj}]_j = [z_{pj}]_j$ in $\mathbb{P}^n$.

- Claim 2 : $\tau_{m,n} \circ \sigma_{m,n} = id_{\mathbb{P}^m \times \mathbb{P}^n}$

Pf/ We prove this on the set $U_k \times V_\ell$. Note $\sigma_{m,n}(U_k \times U_\ell) \subseteq U_{k,\ell}$

$\tau_{m,n} \circ \sigma_{m,n}|_{U_k \times U_\ell} ([x],[y]) = \tau^{(k,\ell)}_{m,n}([x_i y_j]_{i,j}) = ([x_i y_\ell]_i, [x_k y_j]_j) = ([x_i],[y_j])$

because $y_\ell, x_k \neq 0$

- Claim 3 : $\sigma_{m,n} \circ \tau_{m,n} = id_V$ ($\Rightarrow$ im $\sigma_{m,n} = V$)

Pf/ We prove this on the set $V \cap U_{k,\ell}$. Note: $\tau^{(k,\ell)}_{m,n}(V \cap U_{k,\ell}) \subseteq U_k \times U_\ell$

$\sigma_{m,n} \circ \tau_{m,n}|_{V \cap U_{k\ell}} ([z]) = \sigma_{m,n}(([z_{i\ell}]_i, [z_{kj}]_j)) = [z_{i\ell} z_{kj}]_{ij}$

On $V \cap U_{k\ell}$ we have $z_{i\ell} z_{kj} = z_{ij} z_{k\ell}$ for $i \neq k$ & $\ell \neq j$

If $j = \ell$ $z_{i\ell} z_{kj} = z_{ij} z_{k\ell}$ & if $i = k$ then $z_{i\ell} z_{kj} = z_{ij} z_{k\ell}$

In all cases, $z_{i\ell} z_{kj} = z_{ij} \underbrace{\boxed{z_{k\ell}}}_{\neq 0 \text{ on } V\cap U_{k\ell}}$, thus $[z_{i\ell} z_{kj}]_{i,j} = [\underline{z}] \in \mathbb{P}^{mn+m+n}$, as we wanted

$\square$

- **Claim 4:** V is irreducible

Pf/ It's enough to show $V \cap U_{k\ell}$ is irreducible $\forall k,\ell$. In turn

$V \cap U_{k\ell} \xrightarrow[\sigma_{m,n}^{(k,\ell)}]{\sim} U_k \times U_\ell \simeq \mathbb{A}^n \times \mathbb{A}^m$ (see HWS) & $\sigma_{m,n}^{(k,\ell)}$ is an isomorphism of affine varieties. Since $\mathbb{A}^n \times \mathbb{A}^m = \mathbb{A}^{n+m}$ is irreducible, then so is $V \cap U_{k\ell}$.

$\square$

Corollary 1: A product of projective spaces is a projective variety.

Q: Can we cover $\mathbb{P}^m \times \mathbb{P}^n$ with affine spaces?

A: YES! $\mathbb{P}^m \times \mathbb{P}^n = \bigcup_{i,j} U_i \times U_j \qquad U_i \times U_j \cong \mathbb{A}^m \times \mathbb{A}^n$

Note $U_i \times U_j = \sigma_{m,n}(U_{ij} \cap V) = \sigma_{m,n}^{-1}(U_{ij}) \subseteq \mathbb{P}^m \times \mathbb{P}^n$ for $U_{ij} = (i,j)$-standard affine patch on $\mathbb{P}^{mn+m+n}$. Thus, these sets are Zariski open in $\mathbb{P}^m \times \mathbb{P}^n$.

§3 Product of projective varieties:

Q: Can we say the same about a product of projective subvarieties?

A: YES! Use the Segre embedding

Proposition 2: Let $V \subseteq \mathbb{P}^m$ & $W \subseteq \mathbb{P}^n$ be projective varieties with $V = V_{proj}(I)$ & $W = V_{proj}(J)$ for $I \subseteq \mathbb{K}[x_0,\ldots,x_m]$, $J \subseteq \mathbb{K}[y_0,\ldots,y_n]$ homogeneous ideals. Then $V \times W$ is a projective variety. Furthermore:

$$V \times W = V_{proj}\big(\langle f(\tau_0(z)) : f \in I\rangle + \langle g(\tau_1(z)) : g \in J\rangle + \langle z_{ij} z_{k\ell} - z_{i\ell} z_{kj} : i \neq k, j \neq \ell\rangle\big) \subseteq \mathbb{P}^{mn+m+n}$$

Here: $\tau_{m,n} = (\tau_0, \tau_1) : \operatorname{im} \sigma_{m,n} \longrightarrow \mathbb{P}^m \times \mathbb{P}^n$

Proof: Use $\tau_{m,n}$ is a bijection & $V \times W \subseteq \operatorname{im} \sigma_{m,n} \iff \tau_0(z) \in V$ & $\tau_1(z) \in W$ & $z \in \operatorname{im} \sigma_{m,n}$. This happens $\iff$ $f(\tau_0(z)) = 0 \quad \forall f \in I$

$g(\tau_0(z)) = 0 \quad \forall g \in J$

$z \in V_{proj}(z_{ij} z_{k\ell} - z_{i\ell} z_{kj} \quad \substack{i \neq k \\ j \neq \ell})$ $\square$

⚠ The topology on $V \times W$ is **NOT** the product topology of the Zariski topologies on $V \times W$.

Definition: $H(\underline{x}, \underline{y}) \in \mathbb{K}[x_0, \dots x_m, y_0, \dots y_m]$ is **bihomogeneous** of bidegree (d_0, d_1) if $H(x,y) \in \mathbb{K}[y_0, \dots, y_m][x_0, \dots, x_m]$ is homogeneous (in the $\underline{x}$ variables) of degree d_0 & $H(x,y) \in \mathbb{K}[x_0, \dots, x_m][y_0, \dots, y_m]$ is homogeneous (in the y variables) of degree d_1.

Examples (1) $F = x_0 y_0 + x_1 y_1 + x_2 y_2$ in $\mathbb{K}[x_0, x_1, x_2, y_0, y_1, y_2]$ is bihomogeneous of bidegree $(1,1)$

(2) $F = x_0^2 y_1^3 - 2 x_0 x_1 y_0 y_2^2$ in $\mathbb{K}[x_0, x_1, y_0, y_1, y_2]$ is bihomogeneous of bidegree $(2,3)$.

(3) $F = x_0^2 + y_0^3$ is not bihomogeneous.

Corollary 2: Zariski closed sets in $\mathbb{P}^m \times \mathbb{P}^n$ are vanishing loci of bihomogeneous polynomials in $\mathbb{K}[x_0, \dots, x_m, y_0, \dots, y_n]$.

Proof: $Z \subseteq \mathbb{P}^m \times \mathbb{P}^n$ is closed $\iff$ $\sigma_{m,n}(Z) \subseteq \mathbb{P}^{mn+m+n}$ is closed

$\iff \sigma_{m,n}(Z) \cap U_{ij} \subseteq U_{ij} \simeq \mathbb{A}^{mn+m+n}$ is closed

Thm 1 §19.1

$\iff Z \cap \sigma_{m,n}^{-1}(U_{ij}) \subseteq \sigma_{m,n}^{-1}(U_{ij}) = U_i \times U_j \simeq \mathbb{A}^m \times \mathbb{A}^n \simeq \mathbb{A}^{m+n}$

$\sigma_{m,n}$ is cont

is Zariski closed.

Thus $Z \cap \sigma_{m,n}^{-1}(U_{ij}) = V(\langle S \rangle)$ for $S \subseteq \mathbb{K}[\frac{x_0}{x_i}, \dots, \frac{x_m}{x_i}, \frac{y_0}{y_j}, \dots \frac{y_n}{y_j}] \; \forall i,j$

Homogenizing each element of $\langle S \rangle$ with respect to the variables x_i & y_j separately ensures that $Z = V(\langle S\rangle^h)$ where $\langle S\rangle^h \subseteq \mathbb{K}[x_0,\dots,x_m,y_0,\dots,y_n]$ is generated by a collection of bi-homogeneous polynomials.

Lemma 2: In Corollary 2, we can pick all bihomogeneous polynomials to have the same bidegree.

Proof: Set $Z = V(f_1,\dots,f_r)$ with bidegree $(f_i) = (d_i, e_i)$

Set $D = \max\{d_1,\dots,d_r\}$, $E = \max\{e_1,\dots,e_r\}$

Then, $V(f_i) = V\left(\left\{x_k^{D-d_i}\, y_\ell^{E-e_i}\, f_i \ : \ \begin{array}{l} k=0,\dots,m \\ \ell=0,\dots,n \end{array}\right\}\right)$ for all i

so $Z = V\left(\left\{g_{i,k,\ell} := x_k^{D-d_i}\, y_\ell^{E-e_i}\, f_i \quad \begin{array}{l} k=0,\dots,m \\ \ell=0,\dots,n \end{array} \quad i=0,\dots,r\right\}\right)$ &

bidegree $g_{i,k,\ell} = (D-d_i+\deg_x f_i,\ E-e_i+\deg_y f_i) = (D,E)$ $\forall i,k,\ell$

$\square$

Q2: How can we define a regular/rational map $\mathbb{P}^n \times \mathbb{P}^m \longrightarrow \mathbb{P}^N$?

A2: We use the Segre embedding & the standard open cover on $\mathbb{P}^m \times \mathbb{P}^n$

Definition:

$$\begin{tikzcd} \mathbb{P}^m \times \mathbb{P}^n \arrow[r, "\varphi"] \arrow[d, "\wr" ', "\sigma_{m,n}"] & \mathbb{P}^N \\ \mathbb{P}^{mn+m+n} \supseteq V \arrow[ur, "\psi"'] & \end{tikzcd}$$

φ is rational/regular if, and only if $\psi = \varphi \circ \sigma_{m,n}^{-1} : V \longrightarrow \mathbb{P}^N$ is rational or regular.

In turn, ψ is rational/regular if, and only if the following 2 conditions hold:

① [rational maps on affine patches of V]

$$\psi_{ij} = \psi|_{V\cap U_{ij}} : \underset{\substack{\cap\!\!| \\ U_{ij} \simeq \mathbb{A}^{mn+m+n}}}{V \cap U_{ij}} \longrightarrow \mathbb{P}^N \quad \text{is rational/regular for all } ij$$

$$\Longleftrightarrow \quad \psi_{ij}|_{\psi^{-1}(U_k)} : \underset{\substack{\cap\!\!| \\ \mathbb{A}^{mn+m+n}}}{V\cap U_{ij} \cap \psi^{-1}(U_k)} \longrightarrow U_k \simeq \mathbb{A}^N \quad \text{is rational/regular map } \forall k$$

Note: $\{\psi^{-1}(U_k) \cap V \cap U_{ij}\}_k$ is an open cover of the affine variety $V \cap U_{ij} \subseteq U_{ij}$

② [Agreement on overlaps] $\psi_{ij}|_{V\cap U_{ij}\cap U_{k\ell}} = \psi_{k\ell}|_{V\cap U_{ij}\cap U_{k\ell}}$ $\forall i,j,k,\ell$.

Q: Can we give a better characterization for building such maps?

A: Yes, thanks to the fact that $V \cap U_{ij} \simeq U_i \times U_j$ is an affine space & its coordinate ring is $\mathbb{K}[\frac{x_0}{x_i}, \ldots, \frac{x_m}{x_i}, \frac{y_0}{y_j}, \ldots, \frac{y_n}{y_j}]$

Proposition 3: A map $\varphi : \mathbb{P}^m \times \mathbb{P}^n \longrightarrow \mathbb{P}^N$ is rational if, and only if φ corresponds to a list of $N+1$ polynomials $H_0, \ldots, H_N$ in $\mathbb{K}[x_0, \ldots x_m, y_0, \ldots y_n]$ where all $H_k(\underline{x}, \underline{y})$'s are bihomogeneous of the same bidegree

- The map φ is regular if, and only if, $V(H_0, \ldots, H_N) \subseteq \mathbb{P}^m \times \mathbb{P}^n$ is empty.

Proof: By definition φ is rational $\Leftrightarrow$ $\psi : V \longrightarrow \mathbb{P}^N$ is rational

$\Leftrightarrow$ Conditions ① & ② on the previous page hold.

- First we analyze condition ①. Using Proposition 1 §20.1, after clearing denominators, each rational map $\psi_{ij} = V \cap U_{ij} \underset{\sigma_{m,n}|_{U_{ij}}^{-1}}{\simeq} U_i \times U_j = \mathbb{A}^m_{\mathbb{K}} \times \mathbb{A}^n_{\mathbb{K}} \longrightarrow \mathbb{P}^N$

corresponds to a collection of $N+1$ polynomials $P_k^{(ij)}(\frac{x_0}{x_i}, \ldots, \widehat{\frac{x_i}{x_i}}, \ldots \frac{x_m}{x_i}; \frac{y_0}{y_j}, \ldots \frac{y_n}{y_j})$ for $k = 0, \ldots, N$, not all equal 0.

Homogenizing with respect to the variables x_i & y_j separately gives a collection $P_k^{ij} := (P_k^{(ij)})^h \in \mathbb{K}[x_0, \ldots, x_m, y_0, \ldots, y_n]$ where each $P_k^{ij} \equiv 0$ or it is bihomogeneous of bi-degree (d_k^{ij}, e_k^{ij}). For simplicity we say 0 is also bihomogeneous of any arbitrary bi-degree.

- Condition ② will say $[P_k^{ij}]_k = [P_k^{uv}]_k$ in $\mathbb{P}^N$ so we can define φ using $H_k := P_k^{00}$ $\forall k$.

- For the tuple to give a set-theoretic map $\mathbb{P}^m \times \mathbb{P}^n \longrightarrow \mathbb{P}^N$ the entries should have the same bidegree, ie $\forall \lambda, \mu \in \mathbb{K}^*$ we have:

$$[H_0(\lambda \underline{x}, \mu \underline{y}) : \dots : H_N(\lambda \underline{x}, \mu \underline{y})] = [\lambda^{d_0} \mu^{e_0} H_0(\underline{x}, \underline{y}) : \dots : \lambda^{d_N} \mu^{e_N} H_N(\underline{x}, \underline{y})]$$

This is true if, and only if $d_0 = d_1 = \dots = d_N$ & $e_0 = \dots = e_N$, as we wanted.

- For φ to be regular, the map needs to be defined at every point of $\mathbb{P}^m \times \mathbb{P}^n$.

This will happen if, and only if $H_0, \dots, H_N$ have no common solutions in $\mathbb{P}^m \times \mathbb{P}^n$.