Lecture XXIV: Abstract Varieties I

Last Time: we discussed products $X\times Y$ where both X & Y are affine or projective varieties

Q: What happens if we mix? Can we turn $X\times Y$ into an algebro-geometric object?

Say $X \subseteq \mathbb{A}^m$ & $Y \subseteq \mathbb{P}^n$. Then

(1) $X\times Y$ can be covered by a collection of affine varieties

$$Z_i = X\times (Y\cap U_i) \subseteq X\times \mathbb{A}^n \subseteq \mathbb{A}^m\times\mathbb{A}^n \qquad i=0,\ldots,n$$

<u>Note</u>: $Z_i = X\times Y \cap (U_0\times U_i)$ so it should be viewed as an open of $X\times Y$.

($U_0\times U_i$ is an open of $\mathbb{P}^m\times\mathbb{P}^n \xhookrightarrow[\sigma_{m,n}]{} \mathbb{P}^{mn+m+n}$ since $U_0\times U_i = \sigma_{m,n}^{-1}(U_{0i})$)

(2) The opens Z_i, Z_j are related by a change of coordinates on their overlaps.

$$Z_{ij} = Z_i\cap Z_j \subseteq \mathbb{A}^m\times U_{ij} \xrightarrow{\varphi_{ij}} \mathbb{A}^m\times U_{ji} \supseteq Z_j\cap Z_i = Z_{ji}$$
$$(x, [\tfrac{y_k}{y_i}]) \longmapsto (x, [\tfrac{y_k}{y_j}])$$

(3) $\Delta_{X\times Y} = \{((x,y),(x,y)) : (x,y)\in X\times Y\} \subseteq (X\times Y)^2$ is closed

$$= \Delta_{\mathbb{A}^m\times\mathbb{P}^n} \cap \left(V(f_i(\underline{x}), f_i(\underline{x}'), G_j(y), G_j(y'))_{\substack{i=1,\ldots,r\\ j=1,\ldots,s}}\right)$$

if $X = V(f_1,\ldots,f_r)$ $\quad Y = V_{proj}(G_1,\ldots,G_s)$

These 3 conditions will describe what an abstract Variety is. In turn (1) & (2) will define prevarieties.

⚠ The transition functions will allow us to glue the Z_i's together & also glue their sheaves of regular functions $\mathcal{O}_{Z_i}$. (<u>Note</u>: Need to extend our notion of regular functions on irreducible affines to any affine variety.)

<u>Main examples</u>: (1) $\mathbb{P}^n = \bigcup_{i=0}^n U_i$; $U_{ij} \xrightarrow{\varphi_{ij}} U_{ji}$ $\forall i\neq j$;

$\Delta_{\mathbb{P}^n} = V_{proj}(\{x_iy_j - x_jy_i : \substack{i,j=0,\ldots,n\\ i\neq j}\})$ is closed in the projective variety $\mathbb{P}^n\times\mathbb{P}^n \hookrightarrow \mathbb{P}^{mn+m+n}$

(2) $\mathbb{P}^m\times\mathbb{P}^n = \bigcup_{i,j} \underbrace{U_i\times U_j}_{\substack{\text{open in } \mathbb{P}^m\times\mathbb{P}^n\\ \text{by Remark §23.2}}}$ $\qquad U_i\times U_j = \mathbb{A}^n\times\mathbb{A}^m = \mathbb{A}^{m+n}$ is an affine variety

$U_i\times U_j \,\|\, \sigma_{m,n}^{-1}(U_{ij}\subseteq \mathbb{P}^{mn+m+n})$

- The transition functions come in 2 types, using the transition functions from Example (1)

(i) $U_i\times U_j \xrightarrow{(id,\varphi_{jl})} U_i\times U_l$ for $l\neq j$ $\qquad$ (ii) $U_i\times U_j \xrightarrow{(\varphi_{ik}, id)} U_k\times U_l$ for $k\neq i$

(iii) $U_i\times U_j \xrightarrow{(\varphi_{ik},\varphi_{jl})} U_k\times U_l$ for $k\neq i$, $l\neq j$

$\bullet\ \Delta_{\mathbb{P}^m\times\mathbb{P}^n} = V_{proj}\left(x_i x'_k - x_k x'_i,\ y_j y'_\ell - y_\ell y'_j \quad {i\neq k \atop j\neq \ell}\right) \subseteq (\mathbb{P}^m\times\mathbb{P}^n)^2$ is closed

§1. Prevarieties:

Definition: A <u>ringed space</u> is a pair $(X, \mathcal{O}_X)$ where X is a Topological space & $\mathcal{O}_X$ is a sheaf of rings. All our sheaves will satisfy: $\mathcal{O}_X(U) \subseteq \{f : U \longrightarrow \mathbb{A}^1 \text{ continuous}\}$

Definition: A morphism of ringed spaces is a map $(X,\mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ consisting of

① a continuous map $f : X \longrightarrow Y$

② a map of sheaves $f^\# : \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$ where $f_*\mathcal{O}_X$ is the direct image sheaf, defined as $U \longmapsto f_*\mathcal{O}_X(U) = \mathcal{O}_X(f^{-1}(U))$ $\forall\, U \subseteq Y$ open.

More concretely, for each $U \subseteq Y$ open we get a map $\mathcal{O}_Y(U) \xrightarrow{f^\#_U} (f_*\mathcal{O}_X)(U)$ that is compatible with the restriction maps: Given $V \subseteq U$ opens in Y we have the commutative diagram.

$$\begin{tikzcd}
\mathcal{O}_Y(U) \arrow[r, "f^\#_U"] \arrow[d, "\rho^Y_{UV}"'] \arrow[dr, phantom, "\circlearrowleft"] & f_*\mathcal{O}_X(U) = \mathcal{O}_X(f^{-1}(U)) \arrow[d, "\rho^X_{f^{-1}(U), f^{-1}(V)}"] \\
\mathcal{O}_Y(V) \arrow[r, "f^\#_U"] & f_*\mathcal{O}_X(V) = \mathcal{O}_X(f^{-1}(V))
\end{tikzcd}$$

Definition: • A <u>prevariety</u> is a ringed space $(X, \mathcal{O}_X)$ that has a finite open cover by affine varieties $(Z_i, \mathcal{O}_{Z_i})$ ($\mathcal{O}_{Z_i}$ = sheaf of regular functions on Z_i, that if $\mathcal{O}_{Z_i}(U) = \{\varphi : U \to \mathbb{A}^1 : \varphi|_{Y_{ij}\cap U} \in \mathcal{O}_{Y_{ij}}(U\cap Y_{ij}) \ \forall\, Y_{ij}$ irreducible component of $Z_i\}$

$U \cap Y_{ij}$ is an open in the irreducible variety Y_{ij}.

• <u>Morphisms</u> of prevarieties are simply morphisms as ringed spaces

• For each open $U \subseteq X$, we refer to $\mathcal{O}_X(U)$ as the set of <u>regular functions</u> on U.

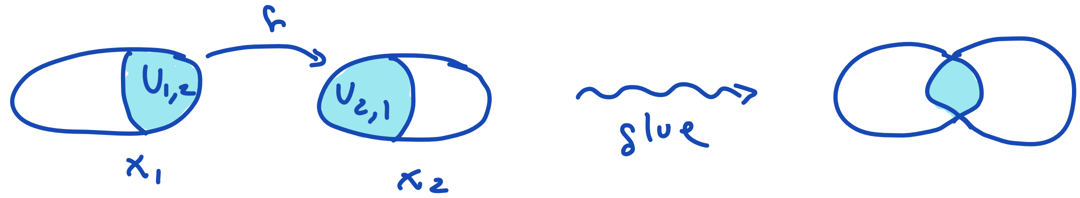
<u>Examples</u>: Affine varieties & open subsets of affine varieties, with the sheaf of regular functions. The finite cover is the trivial one.

• We can build new prevarieties from old ones via gluing. We first discuss the gluing of 2 prevarieties & later do the general case:

Construction 1: Gluing 2 prevarieties.

Fix X_1, X_2 to prevarieties (eg 2 affine varieties) & two opens $U_{1,2} \subseteq X_1$, $U_{2,1} \subseteq X_2$ with an isomorphism $f: U_{1,2} \longrightarrow U_{2,1}$. We define the gluing of X_1 & X_2 along f as $\boxed{X := (X_1 \sqcup X_2)/_\sim}$ with $a \sim a \quad \forall\, a \in X_1 \cup X_2$, $a \sim f(a) \quad \forall\, a \in U_{1,2}$ ($\sim$ is an equiv. reln)

Pictorially:



- We have 2 natural embeddings $i_1 : X_1 \hookrightarrow X \quad i_2 : X_2 \hookrightarrow X$
- We endow X with the quotient topology ($U \subseteq X$ is open $\Leftrightarrow$ $i_1^{-1}(U) \subseteq X_1$ & $U \cap X_2 = i_2^{-1}(U) \subseteq X_2$ are both open. $\quad i_1^{-1}(U) = X_1 \cap U$

Remark: This topology allow us to view X_1, X_2 as open subsets of X. If X_1, X_2 are covered by affine varieties, so is X.

- Sheaf $\mathcal{O}_X$? We obtain it by gluing the sheaves $\mathcal{O}_{X_1}$ & $\mathcal{O}_{X_2}$ along f, as was done in Problem 20 HW4.

regular functions on U $\Leftrightarrow$ restriction to $X_1 \cap U$ lies in $\mathcal{O}_{X_1}(X_1 \cap U)$ &
______ $X_2 \cap U$ ______ $\mathcal{O}_{X_2}(X_2 \cap U)$

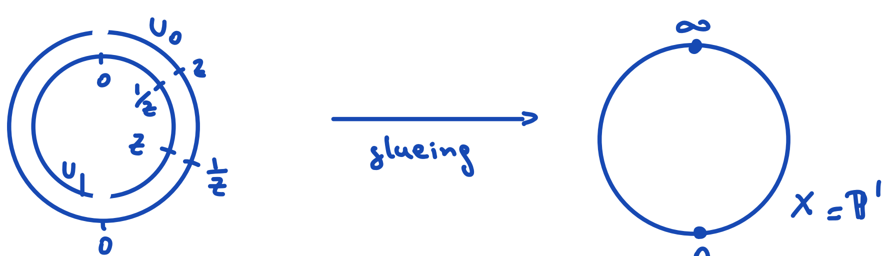
For each $U \subseteq X$ open, we define:

$$\mathcal{O}_X(U) = \{ \varphi : U \longrightarrow \mathbb{K} : \; i_1^* \varphi = \varphi \circ i_1 \in \mathcal{O}_X(i_1^{-1}(U)) \;\&\; i_2^* \varphi = \varphi \circ i_2 \in \mathcal{O}_X(i_2^{-1}(U)) \}$$

(our notion for a regular function)

Examples: ① $\mathbb{P}^1 = U_0 \cup U_1$

Take $\mathbb{A}^1 \supseteq \mathbb{K}^*$, $\mathbb{A}^1 \supseteq \mathbb{K}^*$ glued along $U_{01} = \mathbb{K}^* \xrightarrow{f} \mathbb{K}^* = U_{1,0}$
$z \longmapsto \frac{1}{z}$



$\mathcal{O}_{\mathbb{P}^1}$ = gluing of $\mathcal{O}_{\mathbb{A}^1}$ & $\mathcal{O}_{\mathbb{A}^1}$ along $\mathcal{O}_{\mathbb{A}^1}(\mathbb{K}^*) \longrightarrow \mathcal{O}_{\mathbb{A}^1}(\mathbb{K}^*)$
$\varphi(z)$ reg $\longmapsto \varphi(\frac{1}{z})$ reg.

on $U \subseteq \mathbb{K}^*$: $\frac{f(z)}{g(z)} \longrightarrow \frac{f(\frac{1}{z})}{g(\frac{1}{z})}$ $\quad$ g nowhere vanishing on U & $g(\frac{1}{z})$ nowhere vanishing on $f^{-1}(U)$.

($\Leftrightarrow$ g vanishing at most on $\{p_1, \dots, p_s, \frac{1}{p_1}, \dots, \frac{1}{p_s}\}$)

Using that $\mathbb{K}[x]$ is an UFD we get that g cannot vanish anywhere on $\mathbb{K}^*$.

(2) What if we use $f = id_{\mathbb{A}^1}$?

We get an affine line with 2 zero points.

Q1: Open nbhds of 0 (top)?

A1: We have 2 options: $\begin{cases} V_1 = X \setminus \{p_1, \dots, p_s\} \\ V_2 = X \setminus \{p_1, \dots, p_s, 0^{bot}\} \end{cases}$ $\quad p_1, \dots, p_s \in X \setminus \{0^{top}, 0^{bot}\}$

- $i_1^{-1}(V_1) = \mathbb{A}^1 \setminus \{p_1, \dots, p_s\} = i_2^{-1}(V_1)$ open on $\mathbb{A}^1$
- $i_1^{-1}(V_2) = \mathbb{A}^1 \setminus \{p_1, \dots, p_s\}$; $i_2^{-1}(V_2) = \mathbb{A}^1 \setminus \{p_1, \dots, p_s, 0\}$ open on $\mathbb{A}^1$

Q2: What are $\mathcal{O}_X(V_1)$ & $\mathcal{O}_X(V_2)$?

A2: $\mathcal{O}_X(V_1) = \mathcal{O}_{X_1}(i_1^{-1}(V_1))$; $\mathcal{O}_X(V_2) = \mathcal{O}_{X_1}(i_1^{-1}(V_2))$

Construction 2: Gluing a collection of prevarieties.

Fix $\{(X_i, \mathcal{O}_{X_i})\}_{i \in I}$ prevarieties & subsets $U_{ij} \subseteq X$ for $i \neq j$ with isomorphisms $f_{ij}: U_{ij} \longrightarrow U_{ji}$ satisfying 2 compatibility conditions needed for the gluing:

(1) $f_{ij}^{-1} = f_{ji}$ $\quad \forall i \neq j$

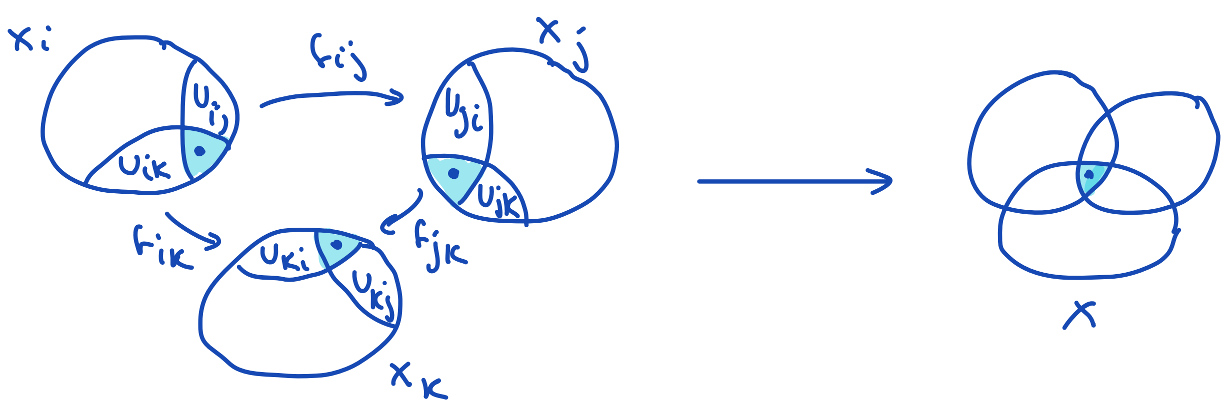
(2) [Cocycle condition] For all i, j, k pairwise distinct

$$f_{ij}^{-1}(U_{ji} \cap U_{jk}) \subseteq U_{ik} \quad \& \quad f_{jk} \circ f_{ij} = f_{ik} \text{ on } f_{ij}^{-1}(U_{jk} \cap U_{ji})$$

Set $\boxed{X := \bigsqcup_i X_i / \sim}$ $\quad$ $a \sim f_{ij}(a)$ $\forall a \in U_{ij} \subseteq X$
$a \sim a$ $\forall a \in X_i$ $\forall i$

Note: $\sim$ is reflexive by construction, symmetric by (1) & transitive by (2)

Pictorially:



- The sheaf $\mathcal{O}_X$ is obtained by gluing the sheaves $\mathcal{O}_{X_i}$ along the maps f_{ij} using Problem 20 HW4. (The condition $V_{ii} = X_i$ & $f_{ii} = id|_{U_{ii}}$ $\forall i$ can be added by hand because we imposed $a \sim a$ $\forall a \in X_i$)

Proposition 1: Prevarieties are Noetherian. In particular, they admit irreducible decompositions of closed subsets into finitely many irreducibles.

Proof: $X = Z_1 \cup \dots \cup Z_r$ Z_i Noetherian (affine variety) $\Rightarrow$ X is Noetherian

Lemma 1: Open subsets of prevariety $(X, \mathcal{O}_X)$ are prevarieties.

Proof: If $U \subseteq X$ is open then $(U, \mathcal{O}_U = \mathcal{O}_X|_U)$ is a ringed space.

- $X = Z_1 \cup \dots \cup Z_r$ with Z_i affine variety, then $U = (Z_1 \cap U) \dots (Z_r \cap U)$ & $(Z_i \cap U) = X_i$ is an open of the affine variety Z_i.
- We can cover X_i as a finite union of basic opens $X_i = \bigcup_{j=1}^{N_i} D(f_{ij})$.

By Problem 1 HW4, $D(f_{ij})$ is an affine variety (indeed, $D(f_{ij}) \overset{cl.}{\hookrightarrow} \mathbb{A}^{n+1}$ if $Z_i \subseteq \mathbb{A}^n$)

So $(X_i, \mathcal{O}_{X_i})$ is a prevariety, covered by $(D(f_{ij}), \mathcal{O}_{D(f_{ij})})$

Conclusion: X is covered by finitely many affine varieties. □

Remark: We call $(U, \mathcal{O}_X|_U)$ an open prevariety of $(X, \mathcal{O}_X)$.

Q: What happens for closed sets?

A: We get a prevariety, but the sheaf is harder to construct!

Definition: Given $Y \subseteq X$ closed set of a prevariety, we define a sheaf $\mathcal{O}_Y$ as those functions that are local restrictions of reg. functions on X, i.e.:

$$\underset{\text{open in } Y}{U} \longmapsto \mathcal{O}_Y(U) := \{ \varphi : U \to \mathbb{K} : \forall a \in U\ \exists\ V \subseteq X \text{ open}, a \in V \ \& \ \psi \in \mathcal{O}_X(V) \text{ with } \varphi = \psi \text{ on } U \cap V \}$$

Lemma 2: $\mathcal{O}_Y$ is a sheaf & $(Y, \mathcal{O}_Y)$ is a prevariety, called a closed prevariety of X

Proof: The sheaf axiom is easy to check because $\mathcal{O}_Y$ is defined by a local property.

For the prevariety claim it is enough to check that for every affine open $U \subseteq X$, the ringed space $(U \cap Y, \mathcal{O}_Y|_{U\cap Y})$ (viewed as an open of Y) is isomorphic to the affine variety $(U\cap Y, \mathcal{O}_{U\cap Y})$, viewed as an affine subvariety of the affine variety U.

By Lemma 1, the latter is a prevariety, so $(Y, \mathcal{O}_Y)$ is a prevariety. (see HW6) $\square$

Corollary 1: If X is a prevariety & $Y \subseteq X$ is closed, then $Y \hookrightarrow X$ is a morphism of prevarieties.

Corollary 2: Fix $f: X \longrightarrow Y$ morphism of prevarieties & $Z \subseteq Y$ open/closed prevariety. Then, $f^{-1}(Z) \subseteq X$ is an open/closed prevariety of X