Lecture XXV: Abstract Varieties II

- **Recall:** $(X, \mathcal{O}_X)$ is a prevariety if it has a finite open cover by affine varieties $\{Z_i\}$ (with the condition $(Z_i, \mathcal{O}_{Z_i})$ satisfies $\mathcal{O}_{Z_i} := \mathcal{O}_X|_{Z_i}$ $(Z_i \subseteq X$ open$)$)
- **Name:** $\mathcal{O}_X$ sheaf of regular functions on X
- **Lemma:** (1) Open & closed subsets of prevarieties are prevarieties & the inclusion map is a morphism of prevarieties

  (2) Prevarieties can be glued along invertible maps on opens $U_{ij} \xrightarrow{f_{ij}} U_{ji}$ satisfying compatibility conditions: $f_{ik} = f_{jk} \circ f_{ij}$ on $U_{ij} \cap U_{ik}$ $\forall i,j,k$) $\quad U_{ij} \subseteq X_i$, $U_{ji} \subseteq X_j$, $f_{ji} = f_{ij}^{-1}$

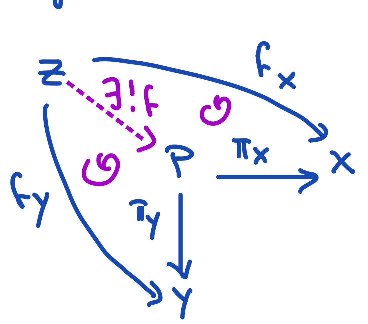
Ex: $\mathbb{P}^1$, ———:———, any affine or projective variety.

§1. Products of Prevarieties:

Next, we show that the product of 2 prevarieties (defined via universal property), is a prevariety.

Remark: We use the universal property as opposed to working with covers $\{U_i\}_{i=1}^{r}$ of X & $\{V_j\}_{j=1}^{s}$ of Y by affine varieties to define $(X \times Y, \mathcal{O}_{X\times Y})$ by the cover $\{U_i \times U_j\}$ to avoid the need to show independence with respect to the choice of covers. The universal property gives uniqueness, & the independence will follow from this.

Definition: Fix X, Y 2 prevarieties. A **product** of X & Y is a prevariety P together with morphisms $\pi_X : P \to X$ & $\pi_Y : P \to Y$ satisfying the following universal property: "given Z prevariety & maps $f_X : Z \to X$, $f_Y : Z \to Y$, then $\exists!\, f : Z \to P$ morphism of prevarieties with $f_X = \pi_X \circ f$ & $f_Y = \pi_Y \circ f$"



Theorem 1: Any 2 prevarieties X & Y have a product. Furthermore, it is unique up to unique isomorphism. We denote it by $X \times Y$. (It's topology is NOT the product Top)

Lemma 1: The statement holds for affine varieties.

Proof: If $X = V(I) \subseteq \mathbb{A}^n$ & $Y = V(J) \subseteq \mathbb{A}^m$, then $P = X \times Y = V(I+J) \subseteq \mathbb{A}^{n+m}$ with $I+J \subseteq \mathbb{K}[\underline{x},\underline{y}]$ ideal. The space P is endowed with the Zariski Topology, NOT the product Topology.

Then: $\mathcal{O}_{X\times Y}(U) = \{ \varphi : U \to \mathbb{A}^1_{\mathbb{K}}$ regular at every point (a,b) in $U \}$
$= \{ \varphi : U \to \mathbb{A}^1_{\mathbb{K}}$ cont $| \ \varphi|_{W\cap U}$ is regular for every irreducible component W of $X \times Y \}$

Locally in an open V of an irreducible component W of $X\times Y$ we have

$$\varphi|_V = \frac{f(x,y)}{g(x,y)} \in \mathbb{K}(W) \quad (f, g \in \mathbb{K}[W]), \ g \text{ nowhere } 0 \text{ on } V$$

The maps $X\times Y \xrightarrow{\pi_X} X$ & $X \times Y \xrightarrow{\pi_Y} Y$ are the usual projections, obtained by restricting $\mathbb{A}^n \times \mathbb{A}^m = \mathbb{A}^{n+m} \to \mathbb{A}^n$ & $\mathbb{A}^n \times \mathbb{A}^m = \mathbb{A}^{n+m} \to \mathbb{A}^m$.

The maps $\mathcal{O}_X \xrightarrow{\pi_X^\#} (\pi_X)_*(\mathcal{O}_{X\times Y})$ & $\mathcal{O}_Y \xrightarrow{\pi_Y^\#} (\pi_Y)_*(\mathcal{O}_{X\times Y})$ have assignments:

On $U \subseteq X$ open: $\varphi \longmapsto \varphi \circ \pi_X|_{U\times Y}$ On $V \subseteq Y$ open: $\varphi \longmapsto \varphi\circ \pi_Y|_{X\times V}$

Since φ is regular & π_X & π_Y are polynomial maps, the compositions are regular maps.

Claim 1: (P, π_X, π_Y) satisfies the universal property.

Pf/ Set-theoretically $P = X\times Y$, so $f_X : Z \to X$, $f_Y : Z \to Y$ determine $f = (f_X, f_Y)$

This function is continuous & satisfies $f_X = \pi_X \circ f$ & $f_Y = \pi_Y \circ f$.
(f is regular $\Leftrightarrow$ f_X & f_Y are regular)

We need to check f is a morphism. We define $f^\# : \mathcal{O}_P \to f_* \mathcal{O}_Z$ on opens $U \subseteq P$: $\mathcal{O}_P(U) \xrightarrow{f^\#_U} \mathcal{O}_Z(f^{-1}(U))$

By the sheaf axiom, its enough to build $f^\#_U(\varphi)$ locally in U so we can work with an open cover $\{U_k\}_k$ of P & $\varphi|_{U_k} = \frac{h_k}{g_k}$ $g_k, h_k \in \mathbb{K}[X\times Y]$ g_k nowhere 0 on V_k..

- $f^\#_U$ corresponds to a pullback map. Indeed: $f^\#(\varphi|_{U_k}) = f^\#\left(\frac{h_k}{g_k}\right) = \frac{f^*(h_k)}{f^*(g_k)}$ & $f^*(h_k), f^*(g_k) \in \mathbb{K}[Z]$ & $f^*(g_k) = g_k \circ f$ is nowhere 0 on $f^{-1}(V) \subseteq Z$.
- By the definition of $\varphi \in \mathcal{O}_P(U)$, the local definition of φ in each open V_k agrees on overlaps. So $\varphi|_{U_k}$ glue to φ.

Thus, by the sheaf axiom, $F^{\#}(\varphi_{|V})$ on V_ℓ's covering the open U glue to a regular map on $F^{-1}(U)$

Claim 2: Uniqueness follows by universal property. (standard argument (Claim 3 in the proof of Theorem 1)) □

Proof of Theorem 1: By Lemma 1, the statement holds for irreducible affine varieties.

- For the general case, take open covers $\{Z_i\}_{i=1}^r$ & $\{V_j\}_{j=1}^s$ of X & Y, respectively, by affine varieties & glue the affine products $Z_i \times V_j$ along the identity morphism, following Construction 2 §24.1. More precisely, $Z_i \times V_j$ & $Z_k \times V_\ell$ are glued along $(Z_i \cap Z_k) \times (V_j \cap V_\ell)$. These sets are open in both $Z_i \times V_j$ & $Z_k \times V_\ell$ because the Zariski topology is finer than the product topology (see Lecture §2.3) & $Z_i \cap Z_k$ is open in X, $V_j \cap V_\ell$ is open in Y, so they are open in Z_i & Z_k, V_j & V_ℓ, respectively.
- We let P be the prevariety obtained by this gluing. The topology of the gluing space P is such that $Z_i \times V_j \hookrightarrow P$ is continuous. In particular $Z_i \times V_j$ is open in P. Thus, P is covered by finitely many opens $Z_i \times V_j$, & all of them are affine varieties.
- By construction, set-theoretically, P is the usual product $X \times Y$.

Claim 1: The maps $Z_i \times V_j \xrightarrow{\pi_1^{(ij)}} Z_i \subseteq X$, $Z_i \times V_j \xrightarrow{\pi_2^{(ij)}} V_j \subseteq Y$ are regular maps between affine varieties, & they glue to morphisms $P \xrightarrow{\pi_X} X$ & $P \xrightarrow{\pi_Y} Y$ via $\pi_X|_{Z_i \times V_j} = \pi_1^{(ij)}$ & $\pi_Y|_{Z_i \times V_j} = \pi_2^{(ij)}$ on $(Z_i \times V_j) \subseteq P$ which cover P.

Pf/ At the topological level, the maps $\pi_1^{(ij)}$, $\pi_2^{(ij)}$ glue to the projections $X \times Y = P \xrightarrow{\pi_1} X$, $X \times Y = P \xrightarrow{\pi_2} Y$.

We prove the claim for sheaves for $\pi_X^{\#}$. The one for $\pi_Y^{\#}$ is analogous.

On each open $U \subseteq X$ we define

$$\pi_X^{\#}(U) : \mathcal{O}_X(U) \longrightarrow (\pi_X)_* \mathcal{O}_P(U \times Y)$$
$$\varphi \longmapsto \varphi \circ \pi_X$$

- φ is obtained by gluing $\varphi_i = \varphi|_{U \cap Z_i}$ = a regular function on the open $U \cap Z_i$ inside the affine variety Z_i using the sheaf axiom.
- Similarly, $\varphi \circ \pi_X$ is obtained by gluing $(\varphi \circ \pi_X)|_{Z_i \cap U \times V_j} = \pi_X^*(\varphi|_{Z_i \cap U})$ $\forall j$ which are regular functions on $(Z_i \cap U) \times V_j$ $\forall j$. The result lies in $(\pi_X)_* \mathcal{O}_P(U \times Y)$. □

Claim 2: (P, π_X, π_Y) satisfies the universal property.

Pf/ Set-theoretically, $P = X \times Y$ so $f_X : Z \longrightarrow P$, $f_Y : Z \longrightarrow P$ determine $f = (f_X, f_Y)$

This function is continuous & satisfies $f_X = \pi_X \circ f$ & $f_Y = \pi_Y \circ f$.

We need to check f is a morphism & we can do this by restricting to an affine open cover of both Z & P by irreducibles that is compatible with f.

Set $\{Z_i \times V_j\}_{i,j}$ to be the cover by products of affine varieties as in the proof of Claim 1.

Consider
$$\boxed{f^{-1}(Z_i \times V_j) = f_X^{-1}(Z_i) \cap f_Y^{-1}(V_j)}$$

By construction, they are open in Z & they cover Z, so they are prevarieties by Lemma 1824.1

- Since f_X & f_Y are morphisms, both $f_X^{-1}(Z_i)$ & $f_Y^{-1}(V_y)$ are prevarieties, so they can be covered by opens which are affine varieties. These intersections are opens in affine varieties, which we cover by basic opens. These are affine varieties.

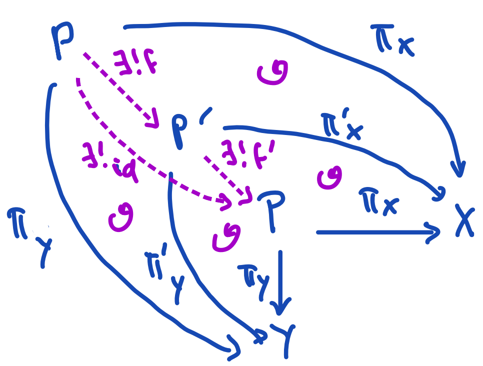
We can break them into their irreducible components.

Conclusion: $Z = W_1 \cup \cdots \cup W_r$ with W_k affine irreducible & $f|_{W_k} : W_k \longrightarrow Z_i \times V_j \subseteq P$

By construction, $f|_{W_k} = (f_X|_{W_k}, f_Y|_{W_k})$ $\quad f_X|_{W_k} : W_k \longrightarrow Z_i$ & $f_Y|_{W_k} : W_k \longrightarrow V_j$ are regular maps, so $f|_{W_k}$ is a regular map. Locally, $f^\#$ corresponds to a pullback map; as we saw in the proof of Lemma 1.

Claim 3: Uniqueness up to unique isomorphism.

Pf/ We let (P', π'_X, π'_Y) be another product & use the universal property



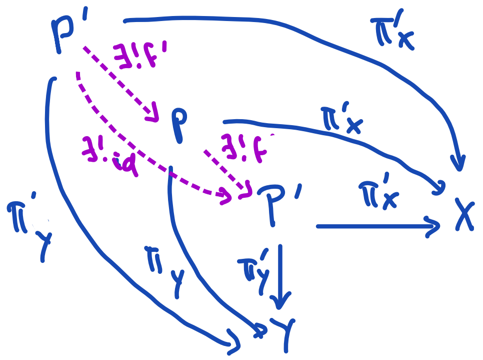
The morphisms $f' \circ f : P \longrightarrow P$ & id_P satisfy the universal prop for P.

$\pi'_X = \pi_X \circ f' \Rightarrow \pi_X = \pi'_X \circ f = (\pi_X \circ f') \circ f = \pi_X \circ (f' \circ f)$

$\pi'_Y = \pi_Y \circ f' \Rightarrow \pi_Y = \pi'_Y \circ f = \pi_Y \circ (f' \circ f)$

So $f' \circ f = id_P$.

Symmetrically:



$f \circ f'$, $id_{P'} : P' \to P'$ satisfy the universal property ($\pi_X = \pi'_X \circ f$ forces $\pi'_X = \pi'_X \circ (f \circ f')$ & $\pi_Y = \pi'_Y \circ f$ forces $\pi'_Y = \pi'_Y \circ (f \circ f')$), so $f \circ f' = id_{P'}$ by uniqueness

□

§2 Varieties:

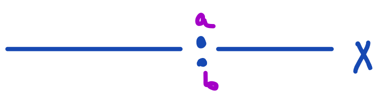
Last time we built  by gluing $\mathbb{A}^1$ with itself along $\mathbb{K}^* \xrightarrow{id} \mathbb{K}^*$. We want to exclude such spaces from consideration. The extra property that we will require is inspired by the following result in general topology:

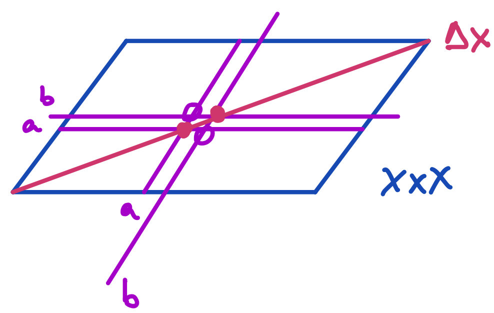
Definition: Given a topological space X, its diagonal $\Delta_X = \{(x,x) : x \in X\}$ is a subset of $X \times X$, endowed with the product topology.

Proposition 1: X is Hausdorff if, and only if $\Delta_X \subseteq X \times X$ is closed.

In our setting the prevariety $X \times X$ is not endowed with the product topology, but the Zariski one, induced by its finite open cover by affine varieties.

Example:





$\mathbb{A}^1 \times \mathbb{A}^1$ with 2 extra lines

Δ_X is not closed : (a,b) $(b,a) \in \overline{\Delta_X} \setminus \Delta_X$

Recall: ① X is the gluing of $\mathbb{A}^1$ with $\mathbb{A}^1$ along $\mathbb{K}^* \xrightarrow{id} \mathbb{K}^*$ $\quad (X = \overset{a}{\overset{\cap}{\mathbb{A}^1}} \cup \overset{b}{\overset{\cap}{\mathbb{A}^1}})$

② $X \times X$ is obtained by gluing 4 copies of $\mathbb{A}^1 \times \mathbb{A}^1$ along $\mathbb{K}^* \times \mathbb{K}^* \xrightarrow{id} \mathbb{K}^* \times \mathbb{K}^*$

so $U \subseteq X \times X$ is open $\iff$ $U \cap (\mathbb{A}^1 \times \mathbb{A}^1) \subseteq (\mathbb{A}^1 \times \mathbb{A}^1) = \mathbb{A}^2$ is open.

Thus
$$U \cap (\mathbb{A}^1 \times \mathbb{A}^1) = \bigcup_{i=1}^{r} D(g_i) \qquad g_i \in \mathbb{K}[x,y].$$
$$= D\left(\prod_{i=1}^{r} g_i\right) = D(g) \qquad \text{for } g \in \mathbb{K}[x,y].$$

<u>Note</u>: The 2 copies of $\mathbb{A}^1 \times \mathbb{A}^1$ agree except at $a \times \mathbb{A}^1$, $b \times \mathbb{A}^1$, $\mathbb{A}^1 \times a$ & $\mathbb{A}^1 \times b$. These are closed subsets of each copy of $\mathbb{A}^1 \times \mathbb{A}^1$

On each copy: $U \cap \mathbb{A}^1 \times \mathbb{A}^1 \cap \Delta_{\mathbb{K}^*} = D(g) \cap \Delta_{\mathbb{K}^*} = \emptyset \iff x - y \mid g$.

But if $(x-y) \mid g$, then $(0,0) \notin U \cap (\mathbb{A}^1 \times \mathbb{A}^1)$ Contra!

Conclusion: $\forall\, U$ open with $(a,b) \in U$ we have $U \cap \Delta_X \neq \emptyset$, thus $(a,b)\ \&\ (b,a) \in \overline{\Delta_X}$
$\forall\, U$ ———— $(b,a) \in U$ ————————

□

Lemma 2: If $X \subseteq \mathbb{A}^n$ is an affine variety, then $X \times X$ is an affine variety in $\mathbb{A}^{2n}$ & $\Delta_X \subseteq X \times X$ is closed in the Zariski Topology.

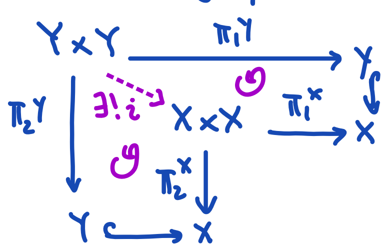
Proof: Since $X \times X \subseteq \mathbb{A}^{2n}$ is closed, it is enough to show $\Delta_X \subseteq \mathbb{A}^{2n}$ is Zariski closed. If $X = V(I) \subseteq \mathbb{A}^n$ then $\Delta_X = V(S) \subseteq \mathbb{A}^{2n}$ where

$$S = \langle \{ f(\underline{x}), f(\underline{y}) \quad f \in I \} \cup \{ x_i - y_i \quad i = 1, \dots, n \} \rangle \subseteq \mathbb{K}[x_1, \dots, x_n, y_1, \dots, y_n]$$

Definition: A prevariety $(X, \mathcal{O}_X)$ is called a <u>variety</u> (or it's <u>separated</u>) if the diagonal $\Delta_X \subseteq X \times X$ is closed.

Lemma 3: Open & closed subsets of varieties are varieties. We will call them open & closed subvarieties, respectively.

Proof: Fix $Y \subseteq X$ open/closed, take the inclusion morphism of prevarieties $i : Y \times Y \longrightarrow X \times Y$, which exists by the universal property of $X \times X$ & $Y \times Y$:



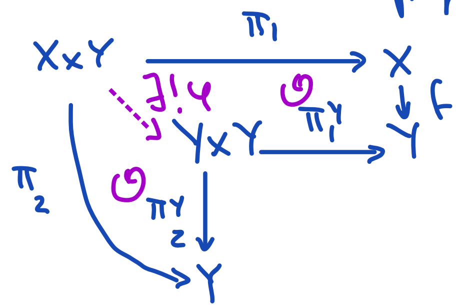
$$\left. \begin{array}{l} \Delta_Y = i^{-1}(\Delta_X) \\ \Delta_X \text{ is closed} \end{array} \right\} \Rightarrow \Delta_Y \subseteq Y \times Y \text{ is closed.}$$
(i continuous!)

Another key property of varieties is the following.

Proposition 2: Fix $f, g : X \longrightarrow Y$ morphisms of prevarieties & assume Y is a variety. Then: (1) The graph $\boxed{\Gamma_f} := \{ (x, f(x)) \ x \in X \} \subseteq X \times Y$ is closed.

(2) The set $\{ x \in X \ \ f(x) = g(x) \} \subseteq X$ is closed.

Proof: (1) We use the universal property of product to build $(f, id) : X \times Y \longrightarrow Y \times Y$

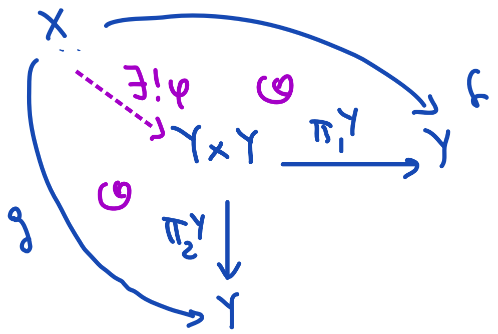


By construction $\varphi = (f, id)$

- Y is a variety, so $\Delta_Y \subseteq Y \times Y$ is closed.

<u>Conclusion:</u> $\Gamma_f = \varphi^{-1}(\Delta_Y) = (f, id)^{-1}(\Delta_Y)$ is closed in $X \times Y$

(2) We proceed in the same fashion:



By construction $\varphi = (f, g)$

- Y is a variety, so $\Delta_Y \subseteq Y \times Y$ is closed.

<u>Conclusion:</u> $\{x : f(x) = g(x)\} = (f, g)^{-1}(\Delta_Y)$, so it is closed in $X \times X$.