Lecture XXVI: Compactness in Algebraic Geometry

§1. Compactness & Hausdorff condition

Definition: A topological space X is sequentially compact if any open cover $\{U_i\}_{i\in I}$ of X admits a finite subcover.

Remark: Any Noetherian topological space is sequentially compact, so affine varieties & abstract prevarieties are sequentially compact.

Definition: A topological space X is compact if and only if for every other topological space Y we have $\pi_2 : X\times Y \longrightarrow Y$, $(x,y)\longmapsto y$ is closed (i.e., for each $Z\subseteq X\times Y$ closed, $\pi_2(Z)\subseteq Y$ is closed). Here, we take the product topology in $X\times Y$.

- The following 2 results are standard statements in general topology.

Proposition 1: X is Hausdorff $\Longleftrightarrow$ $\Delta_X \subseteq X\times X$ is closed (wrt the product Top)

Lemma 1: If X is Hausdorff, then X is compact $\Longleftrightarrow$ X is sequentially compact

⚠ The topology on $X\times Y$ for X, Y prevarieties is NOT the product topology.

More precisely, if $X = X_1\cup\cdots\cup X_r$, $Y = Y_1\cup\cdots\cup Y_s$ are open coverings of X & Y by affine vars, the space $X\times Y$ is obtained by gluing the affine varieties $X_i\times Y_j\subseteq \mathbb{A}^{n_i+m_j}$ $\begin{pmatrix} X\subseteq \mathbb{A}^{n_i}\\ Y\subseteq\mathbb{A}^{m_j}\end{pmatrix}$ endowed with the Zariski Topology along the identity maps on the overlaps.

- The topology on $X\times Y$ is the quotient topology. This implies that the collection $\{X_i\times Y_j\}_{i,j}$ is an open covering of the space $X\times Y$ by affine opens. In particular:

Proposition 2: For X, Y prevarieties we have:

$$Z\subseteq X\times Y \text{ is open/closed} \Longleftrightarrow Z\cap(X_i\times Y_j)\subseteq X_i\times Y_j\subseteq \mathbb{A}^{n_i+m_j} \text{ is Zariski open/closed. } \forall i,j.$$

Consequence: To talk about Algebraic Geometry versions of Hausdorff & compactness we replace the product topology on $X \times X$ & $X \times Y$ by the Zariski-induced one.

- Hausdorff condition $\rightsquigarrow$ Separateness condition (defining varieties vs. prevarieties)
  $\Delta_X \subseteq X \times X$ is Zariski closed
- Compactness condition $\rightsquigarrow$ Completeness condition:

§2 Completeness = compactness in Algebraic Geometry:

Definition: A variety X is called complete if the projection $\pi_2: X \times Y \longrightarrow Y$ is closed for any prevariety Y, when $X \times Y$ is endowed with the Zariski topology.

Lemma 2: It is enough to check the completeness condition on affine varieties Y.

Proof: ($\Rightarrow$) Affine varieties are varieties.

($\Leftarrow$) Assume Y is a prevariety, & write $Y = Y_1 \cup \cdots \cup Y_r$ to be an open covering by affine varieties. Fix $Z \subseteq X \times Y$ closed. We want to show $\pi_2(Z) \subseteq Y$ is closed.

- **Claim 1:** $\pi_2(Z) \subseteq Y$ is closed $\Longleftrightarrow$ $\pi_2(Z) \cap Y_j \subseteq Y_j$ is (Zariski) closed.
  Pf/ $\{Y_j\}_j$'s are an open cover of Y.
- **Claim 2:** $Z \subseteq X \times Y$ closed $\Longleftrightarrow$ $Z \cap (X \times Y_j) \subseteq X \times Y_j$ is closed

Pf/ We consider an open covering $X = X_1 \cup \cdots \cup X_s$ of X affine varieties. Our discussion on page 1 ensures that $X \times Y_j$ is obtained by gluing $X_i \times Y_j$. In particular, by Proposition 2 applied to $X \times Y_j$, we have:

(1) $Z_j := Z \cap (X \times Y_j)$ is closed $\Longleftrightarrow$ $Z_j \cap (X_i \times Y_j) = Z \cap (X_i \times Y_j)$ is Zariski closed

Since $X \times Y$ is obtained by gluing $\{X_i \times Y_j\}_{i,j}$ we have

(2) $Z_j \cap (X_i \times Y_j) = Z \cap (X_i \times Y_j)$ is Zariski closed $\Longleftrightarrow$ $Z \subseteq X \times Y$ is closed

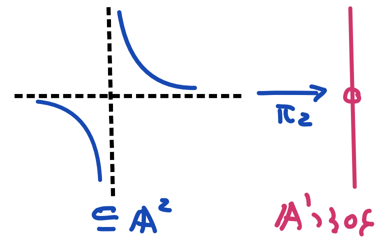
Combining (1) & (2), Claim 2 follows.

- To finish, we note that $\pi_2(Z) \cap Y_j = \pi_2|_{X \times Y_j}(\underbrace{Z \cap (X \times Y_j)}_{\text{closed in } X \times Y_j \text{ by Claim 2}})$

Since Y_j is affine, our hypothesis ensures $\pi_2|_{X \times Y_j}$ is closed. Thus $\pi_2(Z) \cap Y_j \subseteq Y_j$ is closed $\forall j$. Claim 1 ensures that $\pi_2(Z) \subseteq Y$ is closed, as we wanted. $\square$

Remark: An alternative proof of Claim 2 can be given by confirming that $\{X \times Y_j\}_j$ is an open set of $X \times Y$ (indeed, $(X\times Y_j) \cap (X_i \times Y_k) = X_i \times (Y_j \cap Y_k)$ which is open in $X_i \times Y_k$ by construction, precisely because $X\times Y$ is obtained by gluing along the opens $(X_i \times Y_j) \cap (X_k \times Y_\ell) = \underbrace{(X_i \cap X_k)}_{\text{open in } X} \times \underbrace{(Y_j \cap Y_\ell)}_{\text{open in } Y.}$ of both $X_i \times Y_j$ & $Y_k \times Y_\ell$.)

Example: Set $X = Y = \mathbb{A}^1_{\mathbb{K}}$ & consider the projection $\pi_2 : \mathbb{A}^1 \times \mathbb{A}^1 \longrightarrow \mathbb{A}^1$.
Assume that $\mathbb{K}$ is infinite.



The set $Z = V(xy - 1) \subseteq \mathbb{A}^1 \times \mathbb{A}^1$ is Zariski closed. However

$$\pi_2(Z) = \{ y \neq 0 \} = D(y) \subseteq \mathbb{A}^1$$

is open & <u>not</u> closed if $\mathbb{K}$ is infinite

⚠ The issue here is that Z is not closed in the product Topology is $\mathbb{K}$ is infinite

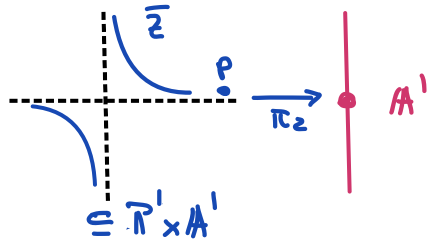
Claim: Given $(x,y) \notin Z$, there exists NO $U \subseteq \mathbb{A}^1$, $V \subseteq \mathbb{A}^1$ Zariski opens with $x \in U$, $y \in V$ & $(U \times V) \cap Z = \emptyset$. Thus Z is <u>not</u> closed in the product Topology of $\mathbb{A}^1 \times \mathbb{A}^1$.

Pf/ Indeed, by construction $U = \mathbb{A}^1 \setminus \{x_1, \ldots, x_s\}$, $V = \mathbb{A}^1 \setminus \{y_1, \ldots, y_r\}$ so

$$U \times V = \mathbb{A}^1 \times \mathbb{A}^1 \setminus \left(\bigcup_{i=1}^{s} (\{x_i\} \times \mathbb{A}^1) \cup \bigcup_{j=1}^{r} (\mathbb{A}^1 \times \{y_j\}) \right)$$

Since $\mathbb{K}$ is infinite, $\exists \; x_0 \in \mathbb{A}^1 \setminus \{x_1, \ldots x_s, y_1, \ldots y_s\}$ with $\frac{1}{x_0} \in \mathbb{A}^1 \setminus \{x_1, \ldots y_s\}$).
In particular, $(x_0, \frac{1}{x_0}) \in (U \times V) \cap Z$.

Example 2: In contrast, take $X = \mathbb{P}^1_{\mathbb{K}}$, $Y = \mathbb{A}^1_{\mathbb{K}}$ with $\mathbb{K}$ infinite. Then
$Z = V(xy - 1) \subseteq \mathbb{A}^1 \times \mathbb{A}^1 \subseteq U_0 \times \mathbb{A}^1 \subseteq \mathbb{P}^1 \times \mathbb{A}^1$ is <u>NOT</u> closed, but
$\overline{Z} = V(x_0 y - x_1) \subseteq \mathbb{P}^1_{[x_0:x_1]} \times \mathbb{A}^1$ is closed $\quad \overline{Z} = Z \cup \{([1:0], 0)\}$
& $\pi_2(\overline{Z}) = \mathbb{A}^1$ is closed in $\mathbb{A}^1$



$\{P\} = \overline{Z} \setminus Z$

Our next goal is to answer the following question:

Q: What's special about projective varieties compared to affine ones?

A: They are "compact" when $\overline{\mathbb{K}} = \mathbb{K}$.

Here is our main result:

Theorem 1: Any projective variety over an algebraically closed field is complete.

We prove the statement for $\mathbb{P}^m$ & $Y = \mathbb{P}^n$ & deduce the result for any X from this, using Lemma 2.

§2 The case of $\mathbb{P}^m$:

Proposition 3: The projection $\pi_2 : \mathbb{P}^m_{\mathbb{K}} \times \mathbb{P}^n_{\mathbb{K}} \longrightarrow \mathbb{P}^n_{\mathbb{K}}$ is closed if $\overline{\mathbb{K}} = \mathbb{K}$

Recall (Corollary 2 §23.3) $Z \subseteq \mathbb{P}^m \times \mathbb{P}^n$ is closed $\Longleftrightarrow Z = V(F_1, \ldots F_r) \subseteq \mathbb{P}^m \times \mathbb{P}^n$ where $F_1, \ldots F_r$ are bihomogeneous polynomials in $\mathbb{K}[x_0, \ldots x_m, y_0, \ldots y_n]$

Lemma 3: We can pick all $F_1, \ldots F_r$ to have the same bidegree.

Example: $r=2$ $n=m=1$, $F_1 = x_0 y_1 + x_1 y_0$ bidegree $(1,1)$ $F_2 = x_0^2 y_0 + x_1^2 y_1$ bidegree $(2,1)$

$V(F_1) = V(F_1 x_0, F_1 x_1)$ & $F_1 x_0, F_1 x_1$ have bidegree $(2,1)$.

Proof: Set $Z = V(F_1 \ldots, F_r)$ with bidegree $(F_i) = (d_i, e_i)$

Set $D = \max\{d_1, \ldots, d_r\}$, $E = \max\{e_1, \ldots, e_r\}$

Then, $V(F_i) = V\left(\{x_k^{D-d_i} y_\ell^{E-e_i} F_i : \begin{array}{l} k=0,\ldots,m \\ \ell = 0, \ldots, n \end{array}\}\right)$ for all i

so $Z = V\left(\{G_{i,k,\ell} := x_k^{D-d_i} y_\ell^{E-e_i} F_i \quad \begin{array}{l} k=0,\ldots m \\ \ell=0,\ldots, n \end{array} \quad i=0,\ldots,r\}\right)$ &

bidegree $G_{i,k,\ell} = (D - d_i + \deg_x F_i, E - e_i + \deg_y F_i) = (D, E)$ $\forall i,k,\ell$ $\square$

Proof: Fix $Z \subseteq \mathbb{P}^m \times \mathbb{P}^n$ closed. By Corollary 2 §23.3 & Lemma 3, we have

$$Z = V(F_1 \ldots F_r) \subseteq \mathbb{P}^n \times \mathbb{P}^m$$

where $F_1, \ldots, F_r$ are bihomogeneous polynomials of the same bidegree (d,e).

To show: $\pi_2(Z)$ is closed in $\mathbb{P}^n$, or equivalently, $\mathbb{P}^n \smallsetminus \pi_2(Z) \subseteq \mathbb{P}^n$ is open.

Next, we fix $a \in \mathbb{P}^m$ & determine polynomial conditions characterizing $a \notin \pi_2(Z)$

. For each $i = 1, \ldots, r$ we define $\boxed{G_i(\underline{x}) = F_i(\underline{x}, \underline{a})} \in \mathbb{K}[x_0, \ldots x_m]$

By construction, each $G_i(x) \in \mathbb{K}[x_0 \ldots x_m]$ is homogeneous of degree d

Since $\overline{\mathbb{K}}=\mathbb{K}$, we can invoke the projective Nullstellensatz:

$$a \notin \pi_2(Z) \iff \nexists\, x \in \mathbb{P}^n \text{ with } (x,a)\in Z \iff V_{proj}(G_1,\ldots,G_r)=\emptyset$$

$$\underset{\text{Nullstellensatz}}{\iff} \sqrt{\langle G_1,\ldots,G_r\rangle} \supseteq I_0 = \langle x_1,\ldots,x_n\rangle$$

$$\iff \exists\, k_1,\ldots,k_r \in \mathbb{Z}_{\geq 1} \text{ with } x_i^{k_i} \in \langle G_1,\ldots,G_r\rangle \quad \forall i$$

$$\iff \mathbb{K}[x_0,\ldots,x_m]_N \subseteq \langle G_1,\ldots,G_r\rangle \text{ for some } N.$$

(for ($\Rightarrow$) Take $N=\sum_{i=1}^{r} k_i$ & for ($\Leftarrow$) take $k_i=N\ \forall i$).

Here, $\mathbb{K}[x_0,\ldots,x_n]_N = N^{th}$ degree component of $\mathbb{K}[x_1,\ldots x_n]$

Note: The last condition can only hold if $N \geq d$. Furthermore, it is equivalent to $\mathbb{K}[x_0,\ldots,x_m]_N = \langle G_1,\ldots,G_r\rangle_N$ (N^{th} graded piece of the homogeneous ideal $\langle G_1,\ldots,G_r\rangle$)

$$= \langle G_1,\ldots,G_r\rangle \cap \mathbb{K}[x_0,\ldots,x_m]_N$$

This says $\forall H \in \mathbb{K}[x_0\ldots x_m]_N\ \exists\, h_1,\ldots,h_r \in \mathbb{K}[x_0,\ldots x_m]$ with

$$\boxed{H = h_1G_1+\cdots+h_rG_r}$$

- Since $H, G_1,\ldots G_r$ are all homogeneous, we can choose all h_i's to be homogeneous with $\deg h_i + \deg G_i = \deg H$, ie $\boxed{\deg H_i = N-d \quad \forall i}$
- This discussion yields a surjective $\mathbb{K}$-linear map:

$$\Phi: \underbrace{\mathbb{K}[x_0,\ldots,x_m]_{N-d}\times\cdots\times\mathbb{K}[x_0,\ldots,x_m]_{N-d}}_{r \text{ times}} \xrightarrow{[G_1,\ldots,G_r]} \mathbb{K}[x_0,\ldots x_m]_N$$

$$(h_1,\ldots,h_r) \longmapsto h_1G_1+\cdots+h_rG_r$$

Let's do some dimension count:

- $\dim((\mathbb{K}[x_0\ldots x_m]_{N-d})^r) = r\dim(\mathbb{K}[x_0,\ldots x_m]_{N-d}) = r\binom{N-d+m}{m}$
- $\dim(\mathbb{K}[x_0,\ldots x_m]_N) = \binom{N+m}{m}$

$\Rightarrow$ The map Φ is represented by a matrix M with $\binom{N+m}{m}$ rows & $r\binom{N-d+m}{m}$ cols

Since the map is surjective, # cols $\geq$ # rows & $\operatorname{rk}(M)=\binom{N+m}{m}$

• If we choose the standard monomial bases on $\mathbb{K}[x_0,\dots,x_m]_{N-d}$ & $\mathbb{K}[x_0,\dots,x_m]_N$ the entries of M are homogeneous polynomials in $\underline{a}$ of degree e (they come from coefficients of $G_i(\underline{x}) = F_i(\underline{x},\underline{a})$ & F_i had bidegree (d,e))

• By construction $rk(M) = \binom{N+m}{m} \leq \#\text{cols } M$ is maximal & this condition is achieved if, and only if one of the maximal minors of A is not vanishing at $\underline{a}$. Such a minor is determined by a choice of columns $(I_1,\dots,I_r)$ with $|I_1 \sqcup \dots \sqcup I_r| = \binom{N+m}{N}$. Each minor is a homogeneous polynomial in $\underline{a} \in \mathbb{P}^n$.

Conclusion: $\underline{a} \notin \pi_2(Z) \iff \underline{a} \in \bigcup_I D(\text{minor}_I) \subseteq \mathbb{P}^n$

Thus, $\pi_2(Z)$ is Zariski closed in $\mathbb{P}^n$. □

Corollary 1: Fix $\overline{\mathbb{K}} = \mathbb{K}$ & let Y be any affine variety. Then, the projection map $\pi_2 : \mathbb{P}^m \times Y \longrightarrow Y$ is closed.

Proof: Assume $Y \subseteq \mathbb{A}^n \simeq U_0 \subseteq \mathbb{P}^n$ & fix a Zariski closed set in $\mathbb{P}^m \times Y$. We consider the closure of Z in the projective variety $\mathbb{P}^m \times \mathbb{P}^n$. By Proposition 1 $\pi_2 : \mathbb{P}^m \times \mathbb{P}^n \longrightarrow \mathbb{P}^n$ is closed, so $\pi_2(\overline{Z}) \subseteq \mathbb{P}^n$ is closed. Now: $Z = \overline{Z} \cap (\mathbb{P}^m \times Y)$ so

$$\pi_2(Z) = \pi_2(\overline{Z} \cap (\mathbb{P}^m \times Y)) = \underbrace{\pi_2(\overline{Z})}_{\text{closed in } \mathbb{P}^n} \cap Y \quad \text{is closed in } Y.$$

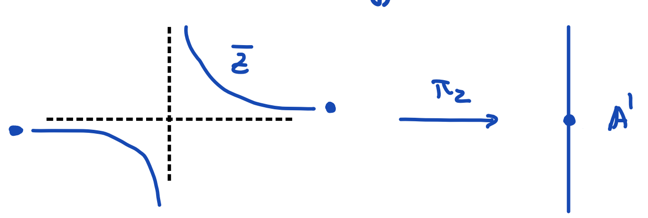
Back to our example on §2:

Example: $\mathbb{A}^1_{\mathbb{K}}$ is not complete because $Z = V(xy-1) \subseteq \mathbb{A}^1 \times \mathbb{A}^1$ is closed but $\pi_2(V(xy-1)) = \mathbb{A}^1 \smallsetminus \{0\} = D(y)$ is not closed. ($\neq \mathbb{A}^1$ & it is not a finite set)

• Consider $Z \subseteq \mathbb{A}^1 \times \mathbb{A}^1 \subseteq \mathbb{P}^1 \times \mathbb{A}^1$ & $\overline{Z} \subseteq \mathbb{P}^1 \times \mathbb{A}^1$, $\overline{Z} = V(x_0 y - x_1)$

Then $\pi_2(\overline{Z}) = \mathbb{A}^1$ ($\pi^{-1}(0) = [1:0]$ & $\pi^{-1}(D_{(y)}) = Z$.)

Adding the points at infinity of Z closes the image.



§3: Proof of Main Theorem:

By Lemma 2, it is enough to check $\pi_2 : X \times Y \longrightarrow Y$ is closed when Y is affine.

Fix $X \subseteq \mathbb{P}^n$ a projective variety & let $Y \subseteq \mathbb{A}^m$ be an affine variety. Fix $Z \subseteq X \times Y \subseteq \mathbb{P}^n \times Y$ a Zariski closed set

Claim 1: $X \times Y$ is closed in $\mathbb{P}^n \times Y$ by construction

Pf/ Since $\mathbb{P}^n = U_0 \cup \cdots \cup U_n$ is an open covering with $U_i \simeq \mathbb{A}^n$ $\forall i$ & Thm 1 §19.1 confirms the Zariski top on $\mathbb{P}^n$ (= closeds sets are $V_{proj}(S)$ for $S \subseteq \mathbb{K}[x_0, \ldots x_n]$ set of homogeneous ideals) agrees with the one making $\mathbb{P}^n$ a prevariety, we have that

$$(\mathbb{P}^m \times Y) \smallsetminus (X \times Y) = (\mathbb{P}^m \smallsetminus X) \times Y \quad \subseteq \mathbb{P}^m \times Y \quad \text{is Zariski open}$$

because $\mathbb{P}^m \smallsetminus X$ is Zariski open in $\mathbb{P}^m$. The later follows since the Zariski topology on $\mathbb{P}^m \times Y$ is finer than the product topology induced by the Zariski top on both $\mathbb{P}^m$ & Y. □

From Claim 1 & Corollary 1, we conclude that $\pi_2(Z) \subseteq Y$ is closed.