Lecture XXVII: Completeness & finite maps I

Recall: X prevariety is complete if $\forall Y$ prevariety we have $\pi_2: X\times Y \longrightarrow Y$ is closed
$(x,y) \longmapsto y$

(Here, $X\times Y$ is endowed with the (Zariski) top induced by the prevariety structure)

Lemma: It is enough to check the case of Y affine varieties (because they are an open cover of any prevariety)

Theorem: If $\overline{\mathbb{K}}=\mathbb{K}$, any projective variety over $\mathbb{K}$ is complete.

Proposition: If $\overline{\mathbb{K}}=\mathbb{K}$ $\pi_2: \mathbb{P}^m\times\mathbb{P}^n \longrightarrow \mathbb{P}^n$ is closed

Remark: Assuming $Z=V(F_1,\ldots,F_r)\subseteq\mathbb{P}^m\times\mathbb{P}^n$ is closed with $F_1,\ldots F_r$ bihomogeneous polynomials in $\mathbb{K}[x_0,\ldots x_m,y_0,\ldots y_n]$, then $\pi_2(Z)\subseteq\mathbb{P}^n$ is closed by the Proposition.
Moreover:

$$\pi_2(Z) = V_{proj}(\langle F_1,\ldots,F_r\rangle\cap\mathbb{K}[\underline{y}])$$

In other words, we need to eliminate the $\underline{x}$ variables from the bihomogeneous ideal generated by $F_1 \ldots F_r$. This can be done using Gröbner bases!

⚠ Not every complete variety is projective (the Hironaka variety, which is a smooth 3-fold)

§1. More on complete varieties:

Here is a key property of complete varieties:

Theorem 1: Let $f: X\longrightarrow Y$ be a morphism of prevarieties. Assume Y is a variety.
If X is complete, then $f(X)$ is a complete closed subvariety of Y.

Proof: Since X is complete, we know that $\pi_2: X\times Y\longrightarrow Y$ is closed.
We prove the statement in 2 parts:

Claim 1: $f(X)\subseteq Y$ is closed, so $f(X)$ is a variety by Lemma 3 §25.2

Pf/ Since $f(X) = \pi_2(\Gamma_f)$ where Γ_f = graph of f in $X\times Y$
$= \{(x,f(x))\ x\in X\}\subseteq X\times Y$,
it suffices to show that Γ_f is closed in $X\times Y$. This is true because Y is a

variety! (Proposition 2(1) §25.2). Indeed, $\Gamma_f = \underbrace{(f\times id)^{-1}}_{\text{cont.}} \underbrace{(\Delta_Y)}_{\text{cl in } Y\times Y} \subseteq X\times Y$ is closed

Claim 2: $f(X) \subseteq Y$ is complete.

Pf/ We check that $f(X)$ satisfies the definition. By Lemma 2 §25.2, it is enough to consider products with affine varieties Y'.

Fix Y' an affine variety & set $\pi_2' : f(X)\times Y' \longrightarrow Y'$. We want to show that $\pi_2'(Z) \subseteq Y$ is closed whenever $Z \subseteq f(X)\times Y'$ is closed.

To do this, we consider the maps:

$$\psi = (f, id) : X\times Y' \longrightarrow Y\times Y' \quad \& \quad \pi_2 : X \times Y' \longrightarrow Y'$$

They are related to π_2' via the commutative diagram:

$$\begin{tikzcd} X\times Y \arrow[r, "\psi"] \arrow[dr, "\pi_2"'] & f(X)\times Y' \arrow[d, "\pi_2'"] \\ & Y \end{tikzcd}$$

Then $\pi_2'(Z) = \pi_2' \circ \psi(\psi^{-1}(Z)) = \pi_2(\underbrace{\psi^{-1}(Z)}_{\text{closed in } X\times Y'})$ is closed in Y' because X is complete. $\square$

Remark: We only need X to be a complete prevariety & Y to be a variety to use the results from §25 & §26.

Corollary 1: If $\overline{\mathbb{K}} = \mathbb{K}$ & X, Y are projective varieties. Then, for any regular map $f : X \longrightarrow Y$ we have that $f(X) \subseteq Y$ is closed (ie, we don't need to take closure of the image to get a projective variety!)

Corollary 2: Assume X is a connected complete prevariety. Then $\mathcal{O}_X(X) = \mathbb{K}$, ie every global regular function on X is constant. In particular, $\mathcal{O}_{\mathbb{P}^n_{\mathbb{K}}}(\mathbb{P}^n_{\mathbb{K}}) = \overline{\mathbb{K}}$

Proof: Pick $\varphi \in \mathcal{O}_X(X)$. We get a regular map $X \xrightarrow{\varphi} \mathbb{A}^1_{\mathbb{K}}$ (locally rational). Clearing denominators will give us a map to $\mathbb{P}^1$, since locally $\varphi|_U = \frac{f}{g}$ & g is nowhere 0 on U.

φ ~~> $\tilde{\varphi} : X \longrightarrow \mathbb{P}^1_{\mathbb{K}}$ Note that $[1:0] \notin \operatorname{Im}\tilde{\varphi}$ so $\operatorname{Im}\tilde{\varphi} \subsetneq \mathbb{P}^1$.

By Theorem 1, $\operatorname{Im}\tilde{\varphi} \subseteq U_0 \subseteq \mathbb{P}^1$ is closed. Thus, $\operatorname{Im}\tilde{\varphi}$ is a finite set (it is proper!)

. φ is continuous & X is connected so $\operatorname{Im}\varphi = \operatorname{Im}\tilde{\varphi}$ is connected.

Conclusion: φ is a pt (connected + finite $\Rightarrow$ a singleton), ie φ is constant.

. When $X = \mathbb{P}^n_{\mathbb{K}}$, Theorem 1 §26.1 ensures X is complete prevariety.

Corollary 3: Let $X \subseteq \mathbb{P}^n_{\mathbb{K}}$ be an irreducible projective variety over $\overline{\mathbb{K}} = \mathbb{K}$ with $|X| \geq 2$. Then, for any homogeneous polynomial $F \in \mathbb{K}[x_0, \ldots x_n]$ of positive degree, we have

$$X \cap V_{proj}(F) \neq \emptyset.$$

Remark: The statement fails for $|X| \leq 1$: given any $p \in \mathbb{P}^n$ we can find F with $p \notin V_{proj}(F)$. The point of the corollary is that we cannot do this if $|X| \geq 2$.

Proof: Pick two distinct pts $p, q \in X$. Given $d = \deg F > 0$ we can find $G \in \mathbb{K}[x_0, \ldots, x_n]_d$ with $G(p) = 0$ & $G(q) \neq 0$ (eg $G(q) = 1$ on a representative of $q \in \mathbb{P}^n$)

(Reason: Finding $G = \sum_{|\alpha| = d} a_\alpha \underline{x}^\alpha$ corresponds to solving a linear system of 2 equations

$$(*) \begin{cases} \sum_{|\alpha| = d} a_\alpha p^\alpha = 0 \\ \sum_{|\alpha| = d} a_\alpha q^\alpha = 1 \end{cases} \quad \text{in } [a_\alpha]_{|\alpha| = d}$$

Since $p \neq q$ and $[p^\alpha]_{|\alpha| = d} = \nu_d(p)$; $[q^\alpha]_{|\alpha| = d} = \nu_d(q)$, the injectivity of ν_d (Theorem 3 §22.2) ensures that the matrix of coefficients of $(*)$ has rank 2 & thus the system is consistent, ie it has a solution)

. We argue by contradiction. Assume $V_{proj}(F) \cap X = \emptyset$, ie $F(x) \neq 0$ $\forall x \in X$.
Thus, the map $\varphi : X \longrightarrow \mathbb{P}^1$ $\varphi(x) = [F(x) : G(x)]$ is regular (F, G are homogeneous of the same degree & F is nowhere 0 in X) & $[0:1] \notin \operatorname{Im}\varphi$

By Theorem 1, $\varphi(X) \subseteq \mathbb{P}^1$ is closed & proper, so $\varphi(X)$ is a finite set of points.
Since X is irreducible, then $\varphi(X)$ is also irreducible (Any decomposition $\varphi(X) = Y_1 \cup Y_2$ gives a decomposition $X = \varphi^{-1}(Y_1) \cup \varphi^{-1}(Y_2)$). Thus, $\varphi(X) = $ pt.
However, $\varphi(p) = [F(p) : G(p)] = [1 : 0]$ & $\varphi(q) = [F(q) : 1]$ so $|\varphi(X)| \geq 2$ Contr! □

Corollary 4: Two curves in $\mathbb{P}^2_{\mathbb{K}}$ intersect.

Remark: This is a particular case of Bézout's Theorem, which will also say how many points of intersection we have!

§2 Finite maps:

In order to set the foundations for developing dimension theory for varieties, we need some preliminaries from Commutative Algebra & the notion of finite maps

Recall: Given 2 commutative rings A, B & a ring homomorphism $\varphi: A \longrightarrow B$ we can view B as an A-module via $a \cdot b := \varphi(a)\, b \in B \quad \forall a \in A, \forall b \in B$

Definition (1) We say φ is finite (or module-finite) if it makes B into a finitely generated A-module, ie $\exists\, b_1, \dots b_r \in B$ s.t

$$B = A\langle b_1, \dots b_r\rangle = \left\{ \sum_{i=1}^{r} \varphi(a_i)\, b_i \;:\; a_1, \dots, a_r \in A \right\}$$

(2) We say φ is of finite type is B is a finitely generated algebra over $\varphi(A)$

(Equivalently, $\exists\; A[x_1, \dots x_n] \xrightarrow{\tilde{\varphi}} B$ surjection extending φ)

(3) We say φ is integral if every $b \in B$ is integral over $\varphi(A)$, ie $\exists\, n \geq 0$ & $a_1, \dots, a_n \in A$ s.t. $P_{(b)}: b^n + \varphi(a_1)\, b^{n-1} + \dots + \varphi(a_{n-1})\, b + \varphi(a_n) = 0$

(Key: $P_{(x)} \in \varphi(A)[x]$ is monic)

(4) An integral extension is an integral map $\varphi: A \longrightarrow B$ that is injective.

Lemma 1: $\varphi: A \longrightarrow B$ is finite $\iff$ φ is integral and of finite type

Proof: For field extensions: $\mathbb{K} \hookrightarrow L$ we know the statement from MATH 6112:

"L is a finite dimensional $\mathbb{K}$-vector space $\iff$ (1) L is algebraic over $\mathbb{K}$ & (2) L is a finitely generated $\mathbb{K}$-algebra"

The proof for modules is verbatim. $\square$

- Our main interest is in the context of affine varieties & regular maps between them.

As we discussed in §11.1, we can always assume our regular maps are dominant.

- Fix $X \subseteq \mathbb{A}^m$, $Y \subseteq \mathbb{A}^n$ (affine) varieties and $\Psi : X \longrightarrow Y$ a dominant regular map.

We have the morphism of sheaves $\Psi^{\#} : \mathcal{O}_Y \longrightarrow \Psi_* \mathcal{O}_X$ on Y induced by pull-back.

In particular, we get a ring homomorphism (in fact a homomorphism of $\mathbb{K}$-algebras)

$$\boxed{\begin{array}{rcl} \varphi = \Psi^{\#}_X : A := \mathcal{O}_Y(Y) & \longrightarrow & \mathcal{O}_X(\Psi^{-1}(Y)) = \mathcal{O}_X(X) =: B \\ f & \longmapsto & \Psi^* f = f \circ \Psi \end{array}}$$

Recall: If X, Y are irreducible and $\overline{\mathbb{K}} = \mathbb{K}$, then by Theorem 1 §13.1 we have

$$\mathcal{O}_X(X) = \mathbb{K}[X] = \frac{\mathbb{K}[x_1, \ldots, x_m]}{I(X)} \quad \& \quad \mathcal{O}_Y(Y) = \mathbb{K}[Y] = \frac{\mathbb{K}[y_1, \ldots, y_n]}{I(Y)}$$

& φ is the pullback map $\begin{array}{rcl} \varphi : A & \longrightarrow & B \\ F & \longmapsto & F \circ \Psi \end{array}$

Our objective is to use finiteness of φ to study the map Ψ. Here is a simple statement:

Theorem: Finite morphisms of varieties have finite fibers (ie $F^{-1}(y)$ is finite for all $y \in Y$.)