Lecture XXVIII: Finite maps II

Recall: Last time we discussed properties of $\varphi: A \longrightarrow B$ ring homomorphism, making B an A-module

- φ finite $\iff$ B is a f.g A-module
- φ finite-type $\iff$ B is a f.g A-algebra
- φ integral $\iff$ every $b \in B$ satisfies a monic polynomial equation in $\varphi(A)[x]$ (B is integral over A)

Lemma: φ is finite $\iff$ φ is of finite-type & integral.

In our context, $(X, \mathcal{O}_X) \xrightarrow{\Psi} (Y, \mathcal{O}_Y)$ morphism of varieties

$\rightsquigarrow$ We get morphisms of $\mathbb{K}$-algebras $\Psi_U^*: \mathcal{O}_Y(U) \longrightarrow \mathcal{O}_X(\Psi^{-1}_{(U)})$ for each $U \subseteq Y$ open.

Definition: A (regular) morphism $\Psi: X \longrightarrow Y$ of affine varieties is <u>finite</u> if $\varphi = \Psi^*: \mathcal{O}_Y(Y) \longrightarrow \mathcal{O}_X(X)$ is a finite homomorphism of $\mathbb{K}$-algebras.

Disclaimer: Throughout, we assume <u>$\mathbb{K}$ is an algebraically closed field</u> since in this context we know $\mathcal{O}_X(X)$. The same result will hold by construction for affine schemes over any ring.

Main Theorem: Any finite map between two affine varieties $/\overline{\mathbb{K}} = \mathbb{K}$ is closed & has finite fibers

§1 The sheaf $\mathcal{O}_X$ for $\overline{\mathbb{K}} = \mathbb{K}$:

By Theorem 1 §13.1 we completely understand $\mathcal{O}_X$ for any irreducible affine variety X.
Indeed: $\mathcal{O}_X(D_X(f)) \cong \mathbb{K}[X]_{(f)}$ $\forall f \in \mathbb{K}[X] \smallsetminus \{0\}$.

- The same statement holds for arbitrary affine varieties:

Theorem 1: Given $X \subseteq \mathbb{A}^n$ affine variety over $\overline{\mathbb{K}} = \mathbb{K}$ and any $p \in X$, we have:
(1) $\mathcal{O}_X(X) \simeq \mathbb{K}[X]$ (2) $\forall f \in \mathbb{K}[X] \smallsetminus \{0\}$, $\mathcal{O}_X(D_X(f)) \simeq \mathbb{K}[X]_{(f)}$ (3) $\mathcal{O}_{X,p} \simeq \mathbb{K}[X]_{m_p}$

Proof: (1) follows from (2) when $f = 1$. (3) follows from (2) since $p \in D_X(\bar{f}) \iff \bar{f}_{(p)} \neq 0 \iff \bar{f} \notin m_p$

We prove (2) Fix $\bar{f} \in \mathbb{K}[X] \smallsetminus \{0\}$, so $\bar{f}_{(x)} \neq 0$ $\forall x \in D_X(f)$. Since $f^n \notin I(X)$ $\forall n \geq 0$ by Hilbert's Nullstellensatz, we conclude that $\{\bar{f}^n\}_{n \geq 0} \subseteq \mathbb{K}[X]$ is multiplicatively closed.

By definition, $\mathcal{O}_X(D_X(f)) = \{\varphi: D_X(f) \longrightarrow \mathbb{A}^1_{\mathbb{K}}$ regular (= locally rational)$\}$.

We get a natural map of $\mathbb{K}$-algebras:

$$\begin{array}{ccc} \mathbb{K}[x_1,\ldots,x_n]_{(f)} & \xrightarrow{\alpha} & \mathcal{O}_X(D_X(f)) \\ \frac{g}{f^m} & \longmapsto & [u \mapsto \frac{g(u)}{f^m}] \end{array} \qquad \forall g \in \mathbb{K}[x].$$

From this point on, the proof is almost verbatim as the one given for the case when X is irreducible (see Lecture 13). The key difference is that localization is not as simple as when $\mathbb{K}[X]$ is a domain.

<u>Claim 1:</u> $\ker(\alpha) = I(X)_{(f)} = I(X)\mathbb{K}[x_1,\ldots x_n]_{(f)}$

Pf/ $\frac{g}{f^m} \in \ker(\alpha) \Leftrightarrow \frac{g}{f^m}(x) = 0 \ \forall x \in D_X(f) \Leftrightarrow g(x) = 0 \ \forall x \in D_X(f). \Leftrightarrow g \in I(D_X(f$

($\supseteq$) is clear because $I(X) \subseteq I(D_X(f))$.

($\subseteq$) If $g \in I(D_X(f)) \Rightarrow gf \in I(X)$ & $\frac{g}{f^m} = \frac{gf}{f^{m+1}} \in I(X)_{(f)}$

<u>Claim 2:</u> α is surjective.

Pf/ Fix $\varphi \in \mathcal{O}_X(D_X(f))$ & opens $V_1,\ldots V_r$ decomposing $D_X(f)$ & $g_1,\ldots,g_r, h_1,\ldots h_r \in \mathbb{K}[x_1\ldots x_n]$ with h_i nowhere 0 on V_i s.t.

$$\varphi_i = \varphi_{|V_i} = \frac{g_i}{h_i} \qquad \forall i=1,\ldots,r$$

. By construction, we can replace V_i by a basic open $V_i = D_X(p_i) \subseteq D_X(h_i)$ for $p_i \notin I(X)$

. <u>Claim 3</u> We may assume $p_i = h_i \ \forall i$

Pf/ $h_i(x) \neq 0 \ \forall x \in D_X(p_i) = X \cap D(p_i) = X \setminus V(p_i) \Rightarrow V_X(h_i) \subseteq V_X(p_i)$

Thus, $p_i \in I(V_X(h_i)) = \sqrt{I(X) + \langle h_i \rangle}$ by Hilbert's Nullstellensatz.

Thus, $\exists\, m \geq 1$ with $p_i^m \in I(X) + \langle h_i \rangle$, ie $\exists\, q \in I(X)$ st $p_i^{m_i} + q \in \langle h_i \rangle$

Since $D_X(\bar{p}_i) = D_X(\bar{p}_i^{m_i}) = D_X(p_i^{m_i} + q)$, we may replace p_i by $p_i^{m_i} + q$ & assume $p_i \in \langle h_i \rangle$.

To finish: $\frac{g_i}{h_i} = \frac{g_i\, p_i/h_i}{h_i\, p_i/h_i} = \frac{(g_i\, p_i/h_i)}{p_i}$ so we can assume $p_i = h_i$, replacing g_i by $g_i\, p_i/h_i$. $\square$

Next, we relate the various rational expressions for φ on each V_i.

• By construction $\frac{g_i}{h_i} = \frac{g_j}{h_j}$ on $D_Y(h_i) \cap D_Y(h_j) = D_Y(h_i h_j)$

By Claim 1 applied to $f = h_i h_j$ we conclude that

$$\frac{g_i}{h_i} = \frac{g_j}{h_j} \quad \text{on} \quad I(X)_{(h_i h_j)} = I(X)\,\mathbb{K}[x_1, \ldots, x_n]_{(h_i h_j)}$$

Thus, $\exists\, N \geq 1$ s.t. $(h_i h_j)^N (g_i h_j - g_j h_i) \in I(X) \quad \forall i, j$

$$\overset{\shortparallel}{g_i h_i^N h_j^{N+1} - g_j h_j^N h_i^{N+1}}$$

(Take N = max of the values for each pair of indices $i \neq j$ in $\{1, \ldots, r\}$)

Replace g_k by $g_k h_k^N$ & h_k by h_k^{N+1} $\forall k = 1, \ldots, r$ to get

$$\frac{g_k}{h_k} = \frac{g_k h_k^N}{h_k^{N+1}} \quad \& \quad g_i h_j - g_j h_i \in I(X) \quad \forall i, j. \quad (*)$$

• Since $D_X(f) = \bigcup_{i=1}^{r} D_X(h_i)$ by construction, we get

$$V_X(f) = X \cap V(f) = \bigcap_{i=1}^{r} X \cap V(h_i) = X \cap V(h_1, \ldots, h_r)$$

Hilbert's Nullstellensatz gives $\sqrt{I(X) + (f)} = \sqrt{I(X) + \langle h_1, \ldots, h_r \rangle}$

In particular, $\exists\, m \geq 1$ & $a_1, \ldots, a_r \in \mathbb{K}[x_1, \ldots, x_n]$ s.t.

$$f^m - \sum_{i=1}^{r} a_i h_i \in I(X)$$

__Claim 5:__ $\varphi = \alpha\left(\frac{a_1 g_1 + \cdots + a_r g_r}{f^m}\right)$

Pf/ $\varphi|_{D_X(h_i)} = \frac{g_i}{h_i} = \frac{a_1 g_1 + \cdots + a_r g_r}{f^m}$ on $D_X(h_i)$ because

$$f^m g_i = \sum_{j=1}^{r} a_j h_j g_i \underset{(*)}{=} \sum_{j=1}^{r} a_j h_i g_j = \Big(\sum_{j=1}^{r} a_j g_j\Big) h_i \mod I(X)$$

So $f^m g_i = h_i \left(\sum_{j=1}^{r} a_j g_j\right)$ on $X \supseteq D_X(h_i)$ $\square$

• A similar statement holds for abstract varieties: $\mathcal{O}_X(U)$ is completely determined by the sets $\mathcal{O}_{X_i}(U \cap X_i)$ for an open covering $X = X_1 \cup \cdots \cup X_r$ by affine opens.

Corollary 1: If $Y \subseteq X \subseteq \mathbb{A}^n$ are affine varieties then $Y \overset{i}{\hookrightarrow} X$ corresponds to $i^\# \; \mathcal{O}_X \longrightarrow i_* \mathcal{O}_Y$. The map $i^\#$ is surjective.

Proof: By construction, $i^\#_X : \mathcal{O}_X(X) = \mathbb{K}[X] \longrightarrow \mathbb{K}[Y] = \mathcal{O}_Y(Y)$ is induced by $I(X) \subseteq I(Y)$. Thus, this map is surjective. On each basic open $D_X(f)$ of X, we have $i^{-1}(D_X(f)) = D_Y(f)$. By Theorem, we have

$$i^\#_{D_X(f)} : \mathcal{O}_X(D_X(f)) = \mathbb{K}[X]_{(f)} \longrightarrow \mathbb{K}[Y]_{(f)} = \mathcal{O}_Y(D_Y(f)) \text{ for every } f \in \mathbb{K}[x_1, \ldots x_n] \smallsetminus I(X).$$

So $i^\#_{D_X(f)}$ is the localization map. Since $i^\#_X$ is surjective, the same is true for $i^\#_X$.

In particular, the map $i^\#_p : \mathcal{O}_{X,p} \longrightarrow \mathcal{O}_{Y,p}$ is surjective $\forall p \in X$. ($\mathcal{O}_{Y,p} = \{0\}$ if $p \notin Y$).

§2 Simplifications on Ψ:

Definition: A morphism $X \xrightarrow{\Psi} Y$ of varieties is a closed immersion if

(1) $\Psi(X) \subseteq Y$ is closed

(2) Ψ induces a homeomorphism between X & $\Psi(X)$

(3) $\Psi^\# : \mathcal{O}_Y \longrightarrow \Psi_* \mathcal{O}_X$ is a surjective morphism of sheaves on Y.

Lemma 1: Any closed subvariety Y of an affine variety X induces a closed immersion $Y \overset{i}{\hookrightarrow} X$.

Proof: By Corollary 1 $\Psi = i^\#_X$ corresponds to the surjection $\mathbb{K}[X] \twoheadrightarrow \mathbb{K}[Y]$ induced by the inclusion $I(X) \subseteq I(Y) \subseteq \mathbb{K}[x_1, \ldots x_n]$ if $X \subseteq \mathbb{A}^n$.

Remark: We can factor Ψ through the closed variety $(\overline{\Psi(X)}, \mathcal{O}_{\overline{\Psi(X)}})$, where $\mathcal{O}_{\overline{\Psi(X)}}$ is the induced subsheaf defined as in Lemma 2 §24.1. By definition, we have:

$$\begin{tikzcd}
(X, \mathcal{O}_X) \arrow[r, "\Psi'"] \arrow[dr, "\Psi"'] & (\overline{\Psi(X)}, \mathcal{O}_{\overline{\Psi(X)}}) \arrow[d, hook, "i"] \\
& (Y, \mathcal{O}_Y)
\end{tikzcd}$$

Lemma 3 §25.2 & Corollary 1 §24.1 ensures that i is a closed embedding

Lemma 2: If $\Psi^\#_Y$ is finite, then $\Psi'_{\overline{\Psi(X)}}$ is finite.

Proof: By construction $\psi_Y^{\#} = (\psi')^{\#}_{\overline{\psi(X)}} \circ i_Y^{\#} : \mathbb{K}[Y] \xrightarrow{i_Y^{\#}} \mathbb{K}[\overline{\psi(X)}] \longrightarrow \mathbb{K}[X]$

$$f \longmapsto f \longmapsto f\circ\psi' = f\circ\psi$$

Since $i_Y^{\#}$ is surjective, it follows that $\mathbb{K}[X]$ is fg over $\mathbb{K}[Y]$ if, and only if, $\mathbb{K}[X]$ is fg over $\mathbb{K}[\overline{\psi(X)}]$. Thus: $\psi_Y^{\#}$ is finite $\Leftrightarrow (\psi')^{\#}_{\overline{\psi(X)}}$ is finite.

Conclusion: We can assume ψ is dominant, by replacing ψ with ψ'.

Q: What is the advantage of working with dominant finite maps $X \xrightarrow{\psi} Y$ between affine varieties $X \subseteq \mathbb{A}^n$ & $Y \subseteq \mathbb{A}^m$ over $\overline{\mathbb{K}} = \mathbb{K}$?

A: Since $\overline{\mathbb{K}} = \mathbb{K}$, Theorem 1 ensures that $\psi_X^{\#}$ corresponds to the pullback map on coord. rings:

$$\boxed{\varphi : A = \mathbb{K}[Y] \longrightarrow \mathbb{K}[X] = B}$$

$A = \mathbb{K}[y_1, \ldots y_n]/_{I(Y)}$

$B = \mathbb{K}[x_1, \ldots x_m]/_{I(X)}$

Furthermore:

Lemma 3: φ is of finite-type & φ is injective (because ψ is dominant)

Proof: Part 1 follows by construction $\varphi|_{\mathbb{K}} = id_{\mathbb{K}}$. Part 2 is a consequence of Problem 9 on HW 4 (ψ dominant $\Leftrightarrow \ker \psi^* \subseteq$ nilradical of $\mathbb{K}[Y]$. But $I(Y)$ is radical by the Nullstellensatz, so $\sqrt{(0)} = \{0\}$ in $\mathbb{K}[Y]$) $\square$

§3. Finite morphisms for $\overline{\mathbb{K}} = \mathbb{K}$: the affine case

Let $X \xrightarrow{\psi} Y$ be a dominant regular map between affine varieties with $X \subseteq \mathbb{A}^m$, $Y \subseteq \mathbb{A}^n$

By Theorem 1, we have the following statement:

Proposition 1: ψ is finite if, and only if, $\psi^{\#}_{D(f)} : \mathcal{O}_Y(D(f)) \longrightarrow \mathcal{O}_X(\psi^{-1}(D(f)))$ is finite $\forall f \in \mathbb{K}[Y] \smallsetminus \{0\}$

Proof: By construction $\mathcal{O}_X(\psi^{-1}(D(f))) = \mathcal{O}_X(D(f\circ\psi))$. By Theorem 1 we conclude that $\psi^{\#}_{D(f)}$ corresponds to the localization of φ: $\psi^{\#}_{D(f)} = \mathbb{K}[Y]_{(f)} \longrightarrow \mathbb{K}[X]_{(f\circ\psi)}$

($\Rightarrow$) φ is finite $\Rightarrow \varphi_{(f)}$ is also finite.

($\Leftarrow$) Take $f = 1$ to conclude $\varphi = \psi^{\#}_{D(1)}$ is finite $\square$

Remark: This will be used to extend the finite definition of morphisms to abstract varieties.

- Next we write some examples of finite morphisms.

Lemma 3: If $X \overset{\psi}{\hookrightarrow} Y$ is a closed immersion of affine vars $/\overline{\mathbb{K}}=\mathbb{K}$, then ψ is finite.

Pf/ It is enough to show that φ is surjective. But this follows by Theorem 1 since $\varphi : \mathbb{K}[Y] \longrightarrow \mathbb{K}[X]$ comes from $I(Y) \subseteq I(X)$. □

Example 1: Fix $Y \subseteq \mathbb{A}^n$ an affine variety $/\overline{\mathbb{K}}=\mathbb{K}$ and $a_1, \ldots, a_m \in \mathcal{O}_Y(Y)$. Set

$$X := \{ (u,t) \in Y \times \mathbb{A}^1 : F_u(t) = t^n + a_{1(u)} t^{n-1} + \cdots + a_{m(u)} = 0 \}$$

Assume $\exists u \in Y$ st $F_u(t) \in \mathbb{K}[t]$ has only simple roots. (*)

Then X is a Zariski closed set of $Y \times \mathbb{A}^1$, and the composition

$$X \overset{i}{\hookrightarrow} Y \times \mathbb{A}^1 \xrightarrow{\pi_1} Y$$

is finite. Furthermore $\mathcal{O}_X(X)$ is a free module over $\mathcal{O}_Y(Y)$ with basis $\{1, t, \ldots, t^{m-1}\}$.

Why? • Using Thm 1 we have $\mathcal{O}_Y(Y) = \mathbb{K}[Y]$ so

$$X = V\left(\langle t^n + A_{1(u)} t^{n-1} + \cdots + A_{m(u)} \rangle + I(Y)\, \mathbb{K}[\underline{u}, y] \right)$$

for $A_i \in \mathbb{K}[x_1, \ldots x_n]$ with $\overline{A_i} = a_i \in \mathbb{K}[u_1, \ldots, u_n]/I(Y)$

Furthermore: $I(X) = \sqrt{I(Y)\, \mathbb{K}[\underline{u}, y] + \langle t^n + A_{1(u)} t^{n-1} + \cdots + A_{m(u)} \rangle}$

$$= I(Y)\, \mathbb{K}[\underline{u}, y] + \langle t^n + A_{1(u)} t^{n-1} + \cdots + A_{m(u)} \rangle$$

The first = arises from the Nullstellensatz & the second one comes from the fact that

$$\frac{\mathbb{K}[u,t]}{(I(Y)\, \mathbb{K}[\underline{u}, y] + \langle t^n + A_{1(u)} t^{n-1} + \cdots + A_{m(u)} \rangle)} = \mathbb{K}[Y][t] \Big/ \langle t^n + A_{1(u)} t^{n-1} + \cdots + A_{0(u)} \rangle$$

has no nilpotents by condition (*).

- $\psi = \pi_1 \circ i$ is finite since $\varphi = \psi_Y^{\#} : \mathbb{K}[Y] \longrightarrow \mathbb{K}[X]$

$$f_{(u)} \longmapsto f \circ \psi = f_{(u)}$$

$$\mathbb{K}[X] = \mathbb{K}[u_1, \ldots u_n, t] \big/ I(Y) + \langle t^n + A_{1(u)} t^{n-1} + \cdots + A_{m(u)} \rangle = \mathbb{K}[Y][t] \big/ \langle t^m + a_{1(u)} t^{m-1} + \cdots a_m \rangle$$

Thus, $\mathbb{K}[Y] \xrightarrow{\varphi} \mathbb{K}[X]$ is of finite-type & integral, hence finite by Lemma 1 §27.2 □

Remark: The condition (*) can be stated by says that the discriminant of $F_u(t)$ does not vanish identically on Y, ie $\operatorname{disc}(F_u(t)) \notin I(Y)$.

• The next result gives a meaning to this terminology:

Theorem 2: A finite morphism $\Psi: X \longrightarrow Y$ between affine varieties over $\overline{\mathbb{K}}=\mathbb{K}$ has finite fibers: $\forall y \in Y$ $\Psi^{-1}(y)$ is finite (possibly empty)

⚠ The size of the fibers can vary along Y. If X & Y are irreducible & Ψ is dominant, this number will be constant on a dense open set of Y.

Proof: By §2 we may assume Ψ is dominant $\quad \Psi^{-1}(y) = \emptyset$ if $y \notin \Psi(X)$

Fix $\mathbb{K}[X] = \mathbb{K}[x_1,\ldots,x_m]/_{I(X)}$, $\quad \mathbb{K}[Y] = \mathbb{K}[y_1,\ldots,y_n]/_{I(Y)}$

By Lemma 3: $\varphi: \mathbb{K}[y_1,\ldots,y_n]/_{I(Y)} \longrightarrow \mathbb{K}[x_1,\ldots x_m]/_{I(X)}$ is of finite type and injective. Thus, since φ is finite, Lemma 1 §27.2 ensures that φ is integral:

$\forall i=0,\ldots,m \quad \exists\ N_i \geq 0$ & $b_{ij} \in \mathbb{K}[Y]$ st $P_i = x^{N_i} + b_{i,1}x^{N_i-1} + \cdots + b_{iN_i} \in \mathbb{K}[Y][x]$ with $P_i(\overline{x}_i) = 0$ in $\mathbb{K}[Y][\overline{x}_i]$

Now, evaluate each $b_{ij} \in \mathbb{K}[Y] = \{Y \to \mathbb{A}^1$ reg$\}$ at $y \in Y$. To get a polynomial $P_{i,y} \in \mathbb{K}[x]$ with $P_{i,y}(\overline{x}_i) = 0$.

Thus: $P_{i,y} \in I(X)$ & $\#\text{Roots}(P_{i,y}) \leq \deg P_{i,y} = N_i$

Claim: $\# \Psi^{-1}(y) \leq (\max\limits_{1\leq i\leq n} N_i)^m$

Pf/ It is enough to check $x \in \Psi^{-1}(y) \Rightarrow P_{i,y}(x_i) = 0$

Indeed $P_{i,y}(x_i) = x_i^{N_i} + b_{i,1}(y)\, x_i^{N_i-1} + \cdots + b_{i,N_i}(y) = 0$

because $P_{i,y} \in I(X)$ & $x \in X$.

So $\Psi^{-1}(y) \subseteq \prod\limits_{i=1}^{m} \{\text{roots of } P_{i,y}\}$ so $|\Psi^{-1}(y)| \leq d^m \quad d = \max\limits_{1\leq i\leq m}\{\deg P_{i,y}\}$

□

⚠ Converse is not true!