Lecture XXIX : Finite maps III

Fix $\overline{\mathbb{K}} = \mathbb{K}$ & a regular morphism $X \xrightarrow{\varphi} Y$ between affine varieties

Definition: We say φ is finite if $\varphi = \varphi^{\#}_{Y} : \mathcal{O}_Y(Y) \longrightarrow \mathcal{O}_X(X)$ is finite
$$\mathcal{O}_Y(Y) = \mathbb{K}[Y], \quad \mathcal{O}_X(X) = \mathbb{K}[X] \quad \text{since } \overline{\mathbb{K}} = \mathbb{K}$$

Lemma: φ finite $\iff \varphi^{\#}_{D(f)} : \mathbb{K}[Y]_{(f)} = \mathcal{O}_Y(D(f)) \longrightarrow \mathcal{O}_X(D(f\circ\varphi)) = \mathbb{K}[X]_{(f\circ\varphi)}$ is finite $\forall f \in \mathbb{K}[Y]\setminus\{0\}$

Remark: φ surjective $\Rightarrow \varphi$ is finite

Example 1: Closed immersion $X \overset{i}{\hookrightarrow} Y$ ((1) $i(X) \subseteq Y$ closed ; (2) $i : X \longrightarrow i(X)$ homeo & (3) $i^{\#} : \mathcal{O}_Y \longrightarrow i_* \mathcal{O}_X$ is a surjective morphism of sheaves on Y)

- We can restrict to dominant regular maps : $X \xrightarrow{\varphi} Y$ finite $\iff X \xrightarrow{\varphi} \overline{\varphi(X)}$ is)

Advantage: φ dominant $\iff \varphi$ is injective.

Theorem: φ has finite fibers

⚠ The size of the fibers can vary along Y. If X & Y are irreducible & φ is dominant, this number will be constant on a dense open set of Y.

⚠ Converse is not true!

Example: Fix $\overline{\mathbb{K}} = \mathbb{K}$ & set $\varphi : V(xy-1) \longrightarrow \mathbb{A}^1$. Then :
$$(x,y) \longmapsto y$$

(1) φ has finite fibers ($|\varphi^{-1}(y)| = 1$ or 0)

(2) φ is <u>not</u> a finite morphism : $\mathbb{K}[x,y]/(xy-1)$ is not a finitely generated $\mathbb{K}[y]$-module ($\mathbb{K}[x] \subseteq \mathbb{K}[x,y]/(xy-1)$)

§1 More on finite maps for affine varieties over $\overline{\mathbb{K}} = \mathbb{K}$:

Lemma 1. If $X \xrightarrow{\varphi} Y$ & $Y \xrightarrow{\eta} Z$ are finite morphisms of affine varieties, then $\eta\circ\varphi : X \longrightarrow Z$ is also finite.

Pf/ On algebras $\mathcal{O}_Z(Z) \xrightarrow{\eta^*} \mathcal{O}_Y(Y)$ are finite & $(\eta\circ\varphi)^* = \varphi^*\circ\eta^*$
$$\mathcal{O}_Y(Y) \xrightarrow{\varphi^*} \mathcal{O}_X(X)$$
makes $\mathcal{O}_X(X)$ a finite $\mathcal{O}_Z(Z)$-module. □

Here is the second main property of finite maps :

Theorem 1: A finite morphism $\psi: X \longrightarrow Y$ between affine varieties over $\overline{\mathbb{K}}=\mathbb{K}$ is closed

To prove this theorem, we will need the following lemmas:

Lemma 2: Given $\psi: X \longrightarrow Y$ a regular map between affine varieties over $\overline{\mathbb{K}}=\mathbb{K}$, write $\varphi = \psi_y^{\#}: \mathbb{K}[Y] \longrightarrow \mathbb{K}[X]$. Fix an ideal $\mathfrak{b} \subseteq \mathcal{O}_Y(Y) = \mathbb{K}[Y]$ & let $Z = V(\mathfrak{b})$.

Then $\psi^{-1}(Z) = V_X(\mathfrak{b}\,\mathcal{O}_X(X)) = V_X(\varphi(\mathfrak{b})\mathbb{K}[X]) = V(\pi^{-1}(\varphi(\mathfrak{b}))) \subseteq X$, where $\pi: \mathbb{K}[x_1, \dots x_n] \twoheadrightarrow \mathbb{K}[X]$ (if $X \subseteq \mathbb{A}^n$) is the natural projection

Proof: $\psi^{-1}(Z) = \{x \in X \mid \underbrace{g(\psi(x))}_{=\varphi(g)(x)} = 0 \;\; \forall g \in \mathfrak{b}\} = V_X(\mathfrak{b}\cdot\mathcal{O}(X)) = V_X(\varphi(\mathfrak{b}))$ $\square$

We'll need the following non-standard version of Nakayama's Lemma:

Nakayama's Lemma: If $A \overset{\varphi}{\hookrightarrow} B$ is finite & $\mathfrak{a} \subseteq A$ is a proper ideal, then $\mathfrak{a}\cdot B \subsetneq B$ is a proper ideal of B.

Proof: The result follows by the standard version of Nakayama's Lemma for f.g. modules over local rings ($(A, \mathfrak{m})$ local ring & M a f.g A-module with $M = \mathfrak{m}\cdot M \Rightarrow M = 0$)

We argue by contradiction & fix $\mathfrak{m} \subseteq A$ maximal ideal with $\mathfrak{a} \subseteq \mathfrak{m}$. We extend B into an $A_{\mathfrak{m}}$-module via $A_{\mathfrak{m}} \xrightarrow{\varphi_{\mathfrak{m}}} (A \setminus \mathfrak{m})^{-1}B \quad \frac{a}{t} \longmapsto \frac{\varphi(a)}{t}$

(If $M_1 \xrightarrow{\psi} M_2$ is a morphism of A-modules & $S \subseteq A$ is multiplicatively closed, then $S^{-1}M_1$, $S^{-1}M_2$ are $S^{-1}A$-modules & ψ extends to $S^{-1}M_1 \xrightarrow{S^{-1}\psi} S^{-1}M_2$ morphism of $S^{-1}A$-modules).

Note: $(A \setminus \mathfrak{m})^{-1}B = \{\frac{a}{t}\cdot b = \frac{\varphi(a)b}{t} \quad a \in A, b \in B \; t \notin \mathfrak{m}\}$ is a ring because the equivalence relation defining $(A \setminus \mathfrak{m})^{-1}B$ is

$$\frac{b}{t} = \frac{b'}{t'} \iff \exists\, t'' \notin \mathfrak{m} \text{ with } \begin{aligned} 0 &= t''\cdot(t'\cdot b - t\cdot b') \\ &= \varphi(t'')(\varphi(t')b - \varphi(t)b') \end{aligned}$$

This ensures that $(A \setminus \mathfrak{m})^{-1}B$ is a ring.

• **Claim 1:** $\varphi_{\mathfrak{m}}$ is injective

Pf/ $\varphi_{\mathfrak{m}}(\frac{a}{t}) = \frac{\varphi(a)}{t} = 0 = \frac{0}{1} \iff \exists\, t' \notin \mathfrak{m}$ with $t''\cdot(\varphi(a)\cdot 1 - 0) = \varphi(t''a) = 0$

But φ is injective, so $t^{n}_{m} a = 0$ in A. Thus, $\frac{a}{t} = \frac{0}{1}$ in A_{m}.

• **Claim 2:** $A_{m}\mathfrak{a} \subseteq A_{m}$ is a proper ideal (since $A_{m}\mathfrak{a} \subseteq mA_{m}$) & $(A \setminus m)^{-1}B$ is a f.g A_{m}-module (can use the same generators of B as an A-module)

$\Rightarrow$ By the standard version of Nakayama's Lemma, we have:

$$A_{m}\mathfrak{a}((A \setminus m)^{-1}B) = A_{m}(\mathfrak{a}B) = A_{m}B = (A\setminus m)^{-1}B \Rightarrow (A\setminus m)^{-1}B = 0$$

However $1_B = \frac{1}{1} \in (A\setminus m)^{-1}B$ & $\frac{1}{1} \neq \frac{0}{1}$ in $(A\setminus m)^{-1}B$ since $t\cdot(1) = \varphi(t) = 0$ forces $t=0$. But $t \notin m$ forces $t \neq 0$. Contradiction. □

Proof of Theorem 1: Pick $Z \subseteq X$ closed & decompose it into irreducibles $Z = Z_1 \cup \cdots \cup Z_r$

Claim 1: If $\psi(Z_i)$ is closed in Y $\forall i$, the same is true for Z.

Claim 2: It suffices to show the statement for X irreducible & $Z = X \subseteq \mathbb{A}^n$.

Pf/ By claim 1 we may focus on the case when $Z \subseteq X$ is irreducible. Since $Z \xhookrightarrow{i} X$ is a closed immersion, then i is finite by Lemma 3 §28.3. Thus, $Z \xrightarrow{\psi \circ i} Y$ is finite by Lemma 1 above, & we are reduced to the case when $Z = X$ □

As discussed in §28.2, we may also assume ψ is dominant. Thus, ψ is closed is equivalent to ψ being surjective. Recall: ψ dominant & finite $\Rightarrow$ φ injective & finite.

Claim 3: ψ is surjective.

Pf/ • Pick $y \in Y$ & let $m_y \subseteq \mathbb{K}[Y]$ be the associated maximal ideal.

• We want to show that $\psi^{-1}(y) \neq \emptyset$

• But $\psi^{-1}(y) \subseteq X$ is closed and $\psi^{-1}(y) = V(\varphi(m_y) + I(X)) \subseteq \mathbb{K}[x_1, \ldots, x_n]$ by Lemma 2.

Thus, by the Nullstellensatz $\psi^{-1}(y) = \emptyset \Leftrightarrow \varphi(m_y) + I(X) = (1)$

By Nakayama's Lemma, we have $m_y \cdot \mathbb{K}[X] \neq \mathbb{K}[X]$. Equivalently, we have
(φ is finite & inj.!)
$\varphi(m_y) + I(X) \neq (1)$. Thus $\psi^{-1}(y) \neq \emptyset$. □

Corollary 1: If $\psi: X \to Y$ is finite & dominant, then ψ is surjective.

⚠ Some textbooks use finite to mean finite as $\varphi = \psi_Y^{\#}$ finite + surjective.

§2. Geometric version of Noether Normalization for $\overline{\mathbb{K}} = \mathbb{K}$:

• Recall Noether Normalization's Theorem for $\mathbb{K}$ infinite seen in Lecture 9

Theorem (NN): Fix an irreducible variety $X \subseteq \mathbb{A}^n$ over $\overline{\mathbb{K}} = \mathbb{K}$. Then, $\exists\ r = 0, \ldots, n$ & $u_1, \ldots, u_r \in \mathbb{K}[X]$ algebraically independent over $\mathbb{K}$, obtained as generic $\mathbb{K}$-linear combinations of $\bar{x}_1, \ldots, \bar{x}_n \in \mathbb{K}[X]$ such that $\mathbb{K}[X]$ is integral over $\mathbb{K}[u_1, \ldots, u_r]$

This allows us to construct another example of a finite map.

Example 2: By NN $\varphi: \mathbb{K}[u_1, \ldots u_r] \hookrightarrow \mathbb{K}[X]$ is finite and

$$u_i = \sum_{j=1}^{n} a_{ij} x_j \quad \forall i = 1, \ldots, r \qquad \text{with} \quad \operatorname{rank}(a_{ij}) = r \quad (A = (a_{ij}) \in \mathbb{K}^{r \times n},\ r \leq n)$$

Up to reordering columns of A we may assume the first $r \times r$ minor of A is invertible on $\mathbb{K}$. Thus, the matrix $\Pi = \left[\begin{array}{c} A \\ \hline 0 \quad I_{n-r} \end{array}\right]$ is invertible over $\mathbb{K}$ and induces an isomorphism $\mathbb{A}^n_{\underline{x}} \xrightarrow{\Pi \cdot} \mathbb{A}^n_{\underline{u}}$

<u>Note</u>: $(\Pi\cdot)^{\#}_{\mathbb{A}^n} : \mathbb{K}[\underline{u}] \longrightarrow \mathbb{K}[\underline{x}]$ becomes $P(\underline{u}) \longmapsto P(A \cdot \underline{x}) \in \mathbb{K}[x_1 \ldots x_n]$

The resulting composition $\psi: X \hookrightarrow \mathbb{A}^n \xrightarrow{\Pi \cdot} \mathbb{A}^n \xrightarrow{\pi_1} \mathbb{A}^r$ where $\pi_1(u_1, \ldots u_N) = (u_1, \ldots, u_r)$ is a finite map. Indeed, by construction

$$\varphi: \mathbb{K}[u_1, \ldots u_r] \longrightarrow \mathbb{K}[u_1, \ldots u_n] \xrightarrow{(\Pi^{\#})_{\mathbb{A}^n}} \mathbb{K}[x_1, \ldots x_n] \longrightarrow \mathbb{K}[X]$$

corresponds to the inclusion $\mathbb{K}[u_1, \ldots u_r] \hookrightarrow \mathbb{K}[X]$ which is finite by construction.

Corollary 2 (NN): Any irreducible affine variety over $\overline{\mathbb{K}} = \mathbb{K}$ admits a finite morphism to some $\mathbb{A}^r$ ($r = \operatorname{trdeg}(\mathbb{K}(X) | \mathbb{K})$) Since φ is injective, ψ is dominant & closed, thus surjective.

• Next we'll see this can be constructed for projective irred vars. To do so, we need to:

(1) extend the finiteness notion for regular maps to abstract varieties over $\overline{\mathbb{K}} = \mathbb{K}$.

(2) prove the corollary with surjectivity condition for projective irreducible varieties.

Note: Corollary 2 will allow us to define $r = \dim X$

§3. Finite morphisms for abstract varieties over $\overline{\mathbb{K}} = \mathbb{K}$:

Fix X, Y two abstract varieties over $\overline{\mathbb{K}} = \mathbb{K}$.

Definition: A morphism $X \xrightarrow{\varphi} Y$ is finite if for every affine open subset $V \subseteq Y$ (ie $V \subseteq Y$ is open & an affine variety), the set $U = \varphi^{-1}(V) \subseteq X$ is an affine variety (with $\mathcal{O}_U = \mathcal{O}_X|_U$) and the induced $\mathbb{K}$-algebra homomorphism $\mathcal{O}_Y(V) \xrightarrow{\varphi = \varphi_V^\#} \mathcal{O}_X(\varphi^{-1}(V))$ is finite (ie, $U \xrightarrow{\varphi|_U} V$ is a finite morphism of affine varieties)

- This definition matches the one given for affine varieties, thanks to the following theorem

Theorem 2: A regular map $\varphi: X \longrightarrow Y$ of abstract varieties is a finite morphism if, and only if, $\exists$ finite open cover of $Y = \bigcup_{j=1}^{s} V_j$ with

① $U_i = \varphi^{-1}(V_i) \subseteq X$ affine $\forall i = 1, \ldots, r$; and

② $\varphi_i := \varphi|_{U_i} : U_i \to V_i$ is finite $\forall i = 1, \ldots, r$

Corollary 3: The definition for affine varieties matches the one for abstract ones over $\overline{\mathbb{K}} = \mathbb{K}$

Pf/ Fix X, Y affine varieties & $X \xrightarrow{\varphi} Y$ finite map between affine varieties. Then, $\mathbb{K}[Y] \xrightarrow{\varphi} \mathbb{K}[X]$ is finite. We use Theorem 2 & verify conditions ① & ② on a suitable affine open cover of Y.

Pick $Y = \bigcup_{i=1}^{n} D(f_i)$ open cover with $f_i \in \mathbb{K}[Y]$. Indeed, such a cover exists because Y is Noetherian & $\{D(f) : f \in \mathbb{K}[Y]\}$ are a basis of opens for the Zariski Topology.

Claim 1: $\varphi^{-1}(D(f_i)) = D(f_i \circ \varphi)$ & $f_i \circ \varphi \in \mathbb{K}[X]$, so $\varphi^{-1}(D(f_i))$ is open in X and an affine variety by Problem 1 HW4. This proves condition ①

Claim 2: $\varphi_i : D_X(f_i \circ \varphi) \longrightarrow D_Y(f_i)$ is finite for all i.

Pf/ This is a consequence of Theorem 1 §28.1: $\mathcal{O}(D_X(f_i \circ \varphi)) = \mathcal{O}(X)_{(f_i \circ \varphi)}$, $\mathcal{O}(D_Y(f_i)) = \mathcal{O}(Y)_{(f_i)}$ $\forall i = 1, \ldots, r$

Each map $\psi^{\#}_{i\,D(f_i)}$ comes from localizing the map φ. Since φ is finite, so is its localization. This proves condition 2. ☐

Once Theorem 2 is proven, we get the statements of finite fibers and closed maps for free from the affine case: Indeed these last conditions can be tested on finite open affine covers.

Corollary 4: Finite regular morphisms between abstract varieties are closed & have finite fibers.