Lecture XXX : Finite maps IV

§1. Finite morphisms for abstract varieties over $\overline{\mathbb{K}}=\mathbb{K}$:

Fix X, Y two abstract varieties over $\overline{\mathbb{K}}=\mathbb{K}$.

Definition: A morphism $X \xrightarrow{\varphi} Y$ is *finite* if for every affine open subset $V \subseteq Y$ (ie $V \subseteq Y$ is open & an affine variety), the set $U = \varphi^{-1}(V) \subseteq X$ is an affine variety (with $\mathcal{O}_U = \mathcal{O}_{X|_U}$) and the induced $\mathbb{K}$-algebra homomorphism $\mathcal{O}_Y(V) \xrightarrow{\varphi=\varphi_V^{\#}} \mathcal{O}_X(\varphi^{-1}(V))$ is finite (ie, $U \xrightarrow{\varphi_{|U}} V$ is a finite morphism of affine varieties)

Lemma 1: Composition of finite maps is finite.

• Our next goal is to prove the following theorem:

Theorem 1: A regular map $\varphi: X \longrightarrow Y$ of abstract varieties is a finite morphism if, and only if, $\exists$ finite open cover of $Y = \bigcup_{j=1}^{s} V_j$ with

① $U_i = \varphi^{-1}(V_i) \subseteq X$ affine $\forall i=1,\ldots,r$; and

② $\varphi_i = \varphi_{|U_i}: U_i \to V_i$ is finite $\forall i=1,\ldots,r$

Corollary 1: Finite morphisms are closed & have finite fibers.

Note: The proof of Theorem 1 uses typical techniques of scheme theory:

(1) Define the property locally ("finite morphisms")

(2) Given 2 affine opens of a cover, describe their intersection by basic opens of both affine opens.

(3) Prove it for some affine open cover $\Rightarrow$ it is true for any cover after refinements.

Lemma 2: Let X be a prevariety over $\overline{\mathbb{K}}=\mathbb{K}$ and $U, V \subseteq X$ affine opens of X. Then, for every $p \in U\cap V$ there exists an open neighborhood $W \subseteq U \cap V$ of p where $W = D_U(f) = D_V(g)$ for some $f \in \mathcal{O}_U(U)$ and $g \in \mathcal{O}_V(V)$ (ie: a basic open of both U & V)

Proof: Since U is affine & $U\cap V$ is open, $\exists\, \tilde{f} \in \mathcal{O}_U(U) = \mathcal{O}_X(U)$ with $p \in D_U(\tilde{f}) \subseteq U \cap V$

Now, $p \in \boxed{W_1 := D_U(\tilde{f})} \subseteq U\cap V \subseteq V$ open, so $\exists\, g \in \mathcal{O}_V(V) = \mathcal{O}_X(V)$ with $\boxed{p \in D_V(g) \subseteq W_1}$

Claim: $W = D_V(g)$ is a basic open in U.

Pf/ • By Theorem 1 §28.1 $\mathcal{O}_X(W_1) = \mathcal{O}_U(W_1) \simeq \mathcal{O}_U(U)_{(\tilde{f})}$ $(\simeq \mathbb{K}[U]_{(\tilde{f})})$

• $g \in \mathcal{O}_X(V)$ & $g|_{W_1} \in \mathcal{O}_X(W_1)$ so $\exists\, h \in \mathcal{O}_X(U)$ & $m \geq 0$ with $g|_{W_1} = \frac{h}{\tilde{f}^m}$ on W_1

Thus, $\boxed{D_V(g) = D_U(\tilde{f}h)}$, so $f = \tilde{f}h \in \mathbb{K}[U]$ works!

Indeed: $D_V(g) = W \subseteq W_1 \subseteq U \cap V \subseteq V$ & $D_U(\tilde{f}h) \subseteq D_U(\tilde{f}) = W_1 \subseteq U \cap V \subseteq U$. So

- $x \in D_V(g) \iff x \in W_1$ & $g(x) \neq 0$
- $x \in D_U(\tilde{f}h) \iff x \in W_1$ & $\tilde{f}h(x) \neq 0 \iff x \in W_1$ & $\tilde{f}(x) \neq 0$ & $h(x) \neq 0$

$\iff x \in W_1$ & $\frac{h}{\tilde{f}^m}(x) \neq 0 \iff x \in W_1$ & $g(x) \neq 0$. $\square$

We'll need the following Proposition:

Proposition 1: Fix a prevariety $X/\overline{\mathbb{K}} = \mathbb{K}$ & $f_1, \ldots f_s \in \mathcal{O}_X(X)$ s.t. $\langle f_1, \ldots f_s \rangle = \mathcal{O}_X(X)$. If $D_X(f_i) = \{x \in X : f_i(x) \neq 0\}$ is an affine variety for all i, then X is affine.

We leave the proof as an optional reading (see Section §3 of these notes).

Note: The statement is true if $X = V(I) \subseteq \mathbb{A}^n$. Indeed:

(1) Using that basic opens are a basis for the Zariski topology, we cover X by finitely many basic opens $D(f_i)$ $\quad X = \bigcup_{i=1}^{s} D(f_i)$ with $f_i \in \mathbb{K}[X]$

(2) The $D(f_i)$'s are affine varieties when viewed in $\mathbb{A}^{n+1}$.

(3) $X = \bigcup_{i=1}^{s} D(f_i) \iff 1 = \langle f_1, \ldots, f_s \rangle \subseteq \mathbb{K}[X]$ by the Nullstellensatz.

Indeed: $X = \bigcup_{i=1}^{s} D(f_i) \iff \emptyset = \bigcap_{i=1}^{s} V(f_i) = V(\langle f_1, \ldots, f_s \rangle)$

$\underset{\text{Nullstellensatz}}{\iff} 1 = \langle f_1, \ldots, f_s \rangle \subseteq \mathbb{K}[X]$

Proof of Theorem 1: We prove the double implication. Set $\varphi_i := (\varphi_i)^{\#}_{V_i}$

($\Rightarrow$) First, we cover $Y = V_1 \cup \cdots \cup V_s$ by open affines of Y. Since f is finite, we know $f^{-1}(V_i) \subseteq X$ is affine & $\mathcal{O}_Y(V_i) \xrightarrow{\varphi_i} \mathcal{O}_X(f^{-1}(V_i))$ is finite.

The sets $\{f^{-1}(V_i)\}_{i=1}^{s}$ are an open cover of X so we are done.

($\Leftarrow$) For this implication, assume we are given a covering $Y = V_1 \cup \cdots \cup V_s$ st $U_i := \varphi^{-1}(V_i) \subseteq X$ is affine open $\forall i$ & $\varphi_i = \varphi|_{U_i} : U_i \longrightarrow V_i$ is a finite regular morphism of affine varieties for each $i = 1, \ldots, r$. That is,

$\varphi_i = \psi^{\#}_{i\,U_i} : U_i \longrightarrow V_i$ is finite.

. Fix $Z \subseteq Y$ affine open. We must show that

① $\psi^{-1}(Z) \subseteq X$ is affine, and

② $\mathcal{O}_X(\psi^{-1}(Z)) \longrightarrow \mathcal{O}_Y(Z)$ is finite.

<u>Claim 1</u>: If $W = D_{V_j}(f) \subseteq V_j$, then $\psi^{-1}(W) = D_{U_j}(f \circ \psi_j) \subseteq U_j$.

Pf/ Since $U_j = \psi^{-1}(V_j)$ is an affine open, $\psi^{-1}(W) \subseteq U_j$ is open & $\varphi_j : \mathbb{K}[V_j] \longrightarrow \mathbb{K}[U_j]$ is the pullback map, we have

$$f \longmapsto f \circ \psi_j$$

$x \in D_{U_j}(f \circ \psi_j) \iff x \in U_j$ & $f \circ \psi_{j(x)} \neq 0 \iff y = \psi(x) \in V_j$ & $f(y) = 0$

$\iff y = \psi(x) \in D_{V_j}(f) \iff x \in \psi^{-1}(D_{V_j}(f))$ □

<u>Claim 2</u>: $\exists$ open affine cover $Z = W_1 \cup \cdots \cup W_r$ where each W_j is a basic open of both Z & V_{j_i} for some $j_i \in \{1, \ldots, s\}$.

Pf/ Since $\underset{\underset{Z}{\cup\!|\,\text{open}}}{Y} = V_1 \cup \cdots \cup V_s$, then $Z = (Z \cap V_1) \cup \cdots \cup (Z \cap V_s)$

Fix $j = 1, \ldots, s$. Since $Z \cap V_j$ is open both in Z & V_j, Lemma 1 ensures that we can cover $Z \cap V_j$ with finitely many opens that are basic (affine) opens of both Z & V_j. Write: $Z \cap V_j = \bigcup_{i=1}^{r_j} W_i^{(j)}$.

Putting these sets together gives a covering of $\boxed{Z = W_1 \cup \cdots \cup W_r}$, by affine opens satisfying the desired properties. □

We write

$$(*)\quad \boxed{\begin{array}{lll} W_i = D_Z(g_i) & \text{for some} & g_1, \ldots, g_r \in \mathbb{K}[Z] \\ W_i = D_{V_{j_i}}(f_i) & \text{———} & f_i \in \mathbb{K}[V_{j_i}] \ \ (\text{if } W_i \subseteq V_{j_i}) \end{array}}$$

<u>Claim 3</u>: $\psi^{-1}(Z)$ is an affine variety. (so condition ① holds)

Pf/ Since $Z = W_1 \cup \cdots \cup W_r = D_Z(g_1) \cup \cdots \cup D_Z(g_r)$ & Z is affine, the Nullstellensatz ensures

$$(1) = \langle g_1, \dots, g_r \rangle \subseteq \mathbb{K}[Z] \underset{\overline{\mathbb{K}}=\mathbb{K}}{=} \mathcal{O}_Z(Z)$$

The map $\psi^{-1}(Z) \xrightarrow{\psi} Z$ yields a map of $\mathbb{K}$-algebras:

$$\mathcal{O}_Z(Z) = \mathbb{K}[Z] \xrightarrow{\alpha} \mathcal{O}_X(\psi^{-1}(Z))$$
$$g_i \longmapsto g_i \circ \psi_{|\psi^{-1}(Z)}$$

Set $\boxed{G_i = g_i \circ \psi \in \mathcal{O}_X(\psi^{-1}(Z))}$ for $i = 1, \dots, r$.

<u>We claim</u>: $\langle G_1, \dots, G_r \rangle = \mathcal{O}_X(\psi^{-1}(Z))$ (ie they generate the unit ideal)

Indeed $1 = \sum_{i=1}^r a_i g_i$ for $a_i \in \mathbb{K}[Z] \Rightarrow 1 = \alpha(1) = \sum_{i=1}^r \alpha(a_i)\,\alpha(g_i)$

$$= \sum_{i=1}^r \underbrace{\alpha(a_i)}_{\in \mathcal{O}_X(\psi^{-1}(Z))} G_i .$$

- In addition, ψ is obtained as the gluing of the maps $\psi^{-1}(W_i) \xrightarrow{\psi_{ji}} W_i$ if $W_i \subseteq V_{ji}$.
- Since $D_{\psi^{-1}(Z)}(g_i \circ \psi) = \{x \in \psi^{-1}(Z) : \underset{(x)}{g_i \circ \psi} \neq 0\} = \psi^{-1}(D_Z(g_i)) = \psi^{-1}(W_i)$

By $(*)$ $\psi^{-1}(W_i) = \psi^{-1}(D_{V_{ji}}(f_i)) \underset{\text{Claim 1}}{=} D_{U_{ji}}(f_i \circ \psi_{|U_{ji}}) \subseteq \underset{\text{affine}}{U_{ji}}$ & $f_i \circ \psi_{|U_{ji}} \in \mathbb{K}[U_{ji}]$

Thus: $D_{\psi^{-1}(Z)}(g_i \circ \psi)$ is an affine variety.

<u>Conclusion</u>: $\{G_1, \dots, G_r\} \subseteq \mathcal{O}_{\psi^{-1}(Z)}(\psi^{-1}(Z))$ satisfy

(1) $\langle G_1, \dots, G_r \rangle = \mathcal{O}_{\psi^{-1}(Z)}(\psi^{-1}(Z)) = \mathcal{O}_X(\psi^{-1}(Z))$

(2) $D_{\psi^{-1}(Z)}(G_i)$ is an affine variety $\forall i$

Thus, by Proposition 1, we conclude that $\psi^{-1}(Z)$ is an affine variety.

<u>Claim 4</u>: $\psi_{|\psi^{-1}(W_i)} : \psi^{-1}(W_i) \longrightarrow W_i \subseteq Z$ is finite for all $i = 1, \dots, r$.

Pf/ By Claims 2 & 1 & $(*)$ $\psi^{-1}(W_i) = D_{U_{ji}}(f_i \circ \psi_{|U_{ji}}) \subseteq U_{ji}$, so it affine (by Problem 1 HW4). Furthermore, since φ_{ji} is finite, Proposition 1 §28.3

ensures that
$$\varphi^{\#}_{\varphi^{-1}(W_i)} = \mathbb{K}[W_i] \longrightarrow \mathbb{K}[\varphi^{-1}(W_i)] \text{ is finite}$$
$$\mathbb{K}[W_i] \overset{\text{by }(*)}{=} \mathbb{K}[V_{ji}]_{(f_i)} = \mathcal{O}_Y(W_i), \qquad \mathbb{K}[\varphi^{-1}(W_i)] \overset{\text{by }(*)}{=} \mathbb{K}[U_{ji}]_{(f_i\circ\varphi_{ji})} = \mathcal{O}_X(\varphi^{-1}(W_i))$$

(it is the localization of a finite morphism) $\square$

<u>**Claim 5:**</u> $\mathcal{O}_Y(Z) \xrightarrow{\varphi} \mathcal{O}_X(\varphi^{-1}(Z))$ is finite.

Pf/ Since Z is affine, $\mathcal{O}_Y(Z) = \mathcal{O}_Z(Z) \simeq \mathbb{K}[Z]$

$\mathcal{O}_X(\varphi^{-1}(Z)) = \mathcal{O}_{\varphi^{-1}(Z)}(\varphi^{-1}(Z)) \simeq \mathbb{K}(\varphi^{-1}(Z)]$.

Furthermore $W_i = D_Z(g_i)$ & $\varphi^{-1}(W_i) = D_{\varphi^{-1}(Z)}(g_i\circ\varphi)$, so Theorem 1

gives $\varphi^{\#}_{|\varphi^{-1}(W_i)}\ \mathcal{O}_Y(Z)_{(g_i)} \longrightarrow \mathcal{O}_X(\varphi^{-1}(Z))_{(g_i\circ\varphi)}$, which is finite by

Claim 3.

Since $\langle g_1,\ldots,g_s\rangle = 1 = \mathbb{K}[Z]$ we conclude by standard commutative algebra arguments (discussed in Lemma 2) that $\varphi : \mathcal{O}_Y(Z) \longrightarrow \mathcal{O}_X[\varphi^{-1}(Z)]$ is finite. $\square$

<u>**Lemma 2:**</u> Let A be a commutative ring & M an A-module. Fix $\{x_1,\ldots,x_n\}\subseteq A$ satisfying (1) x_i is <u>NOT</u> nilpotent for all i.

(2) $\langle x_1,\ldots,x_n\rangle = A$

If $x_i^{-1}M = A_{(x_i)}\cdot M$ is a finitely generated $A_{(x_i)}$-module for all i, then M is a finitely generated A-module.

<u>**Proof:**</u> Since each x_i is not nilpotent, we can consider the ring of fractions

$$A_{(x_i)} = S_i^{-1}A \quad \text{for} \quad S_i = \{1, x_i, x_i^2, \ldots\}$$

Fix $\{m_j^{(i)}\ j=1,\ldots,N_i\} \subseteq M$ generating $A_{(x_i)}M$ over $A_{(x_i)}$, that is

$$A_{(x_i)}\left\langle \frac{m_1^{(i)}}{1},\ldots,\frac{m_{N_i}^{(i)}}{1}\right\rangle = x_i^{-1}M.$$

($m/x_i^r = \frac{1}{x_i^r}\cdot\frac{m}{1}$, so we can always pick a finite generating set in M)

Consider N = A-submodule of M generated by $\{m_j^{(i)} : 1\leq j\leq N_i ; 1\leq i\leq n\}$ & let $M' = M/N$ be the quotient module.

Claim 1: $x_i^{-1} \cdot M' = \{0\}$ $\forall i$, ie, x_i acts nilpotently on M', meaning given $m\in M'$ $\exists\, r_i >> 0$ s.t $x_i^{r_i}.m = 0$)

Pf/ $$A_{(x_i)} M' = S_i^{-1} M' = S_i^{-1}(M/N) = \frac{S_i^{-1}M}{S_i^{-1}N} \underset{\text{by construction of } N}{=} \frac{S_i^{-1}M}{S_i^{-1}M} = 0.$$

Claim 2: $M' = 0$.

Pf/ given any m pick $r_1,\dots,r_n$ st $x_i^{r_i} m = 0$ $\forall i$.

Write $1 = \sum_{i=1}^n a_i x_i$ & set $N := r_1 + \dots + r_n$. Then for every $m\in M'$

$$m = 1\cdot m = \left(\sum_{i=1}^n a_i x_i\right)^N \cdot m \subseteq \langle \underbrace{x_i^{r_i} m}_{=0 \text{ by Claim 1}} \quad i=1,\dots,n\rangle = 0$$

Thus $m = 0$ $\forall m\in M'$. □

§2 Linear Projections:

Recall: Last Time we discussed a geometric version of Noether Normalization for irreducible affine varieties over $\overline{\mathbb{K}} = \mathbb{K}$

Theorem (NN). Given $X\subseteq \mathbb{A}^n$ irreducible affine variety over $\overline{\mathbb{K}}=\mathbb{K}$, there exists $\{u_1,\dots,u_r\}\subseteq \mathbb{K}[X]$ alg indep over $\mathbb{K}$ with $\underline{u} = A\cdot\begin{bmatrix} x_1\\ \vdots \\ x_n\end{bmatrix}$ for some $A\in\mathbb{K}^{r\times n}$ generic, with $\mathbb{K}[u_1,\dots u_r] \xhookrightarrow{\varphi} \mathbb{K}[X]$ integral ($\Rightarrow$ finite).

Geometrically, this corresponds to a finite map $\psi : X \longrightarrow \mathbb{A}^r_{[u_1,\dots u_r]}$

Corollary 2: Let X be an irreducible affine variety. Then, there is a finite surjective morphism $X \longrightarrow \mathbb{A}^r$ for some r.

Proof: By Theorem NN we can build such a finite map & $\varphi = \psi^{\#}_{\mathbb{A}^r}$ is injective. Thus, by Problem 9 HW4, ψ is dominant. Finite maps are closed, so ψ is surjective.

Q: What can we do for projective varieties?

Recall the construction of linear projections from a point $p\in\mathbb{P}^n$.

Pick $H \subseteq \mathbb{P}^n$ any linear hyperplane not containing p. By Problem 9 HW5, $H \simeq \mathbb{P}^{n-1}$

set: $\psi_p : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$, $q \longmapsto L_{pq} \cap H$. This map is rational, not defined at p, but defined on $U = \mathbb{P}^n \setminus \{p\}$ (dense open in $\mathbb{P}^n$)

Note: ψ_p is linear & surjective. This map has fibers of infinite size, so it is NOT finite. However, on H, $\psi_p : H \xrightarrow{\sim} H$ is an iso, hence finite.

Our objective is two-fold:

① Generalize this construction to arbitrary linear spaces of $\mathbb{P}^n$

② Given $X \subseteq \mathbb{P}^n$ irreducible, find a suitable linear space Λ & a projection

$$\pi_\Lambda : \mathbb{P}^n \dashrightarrow \mathbb{P}^r \quad \text{well-defined away from } \Lambda$$

such that $\pi_{\Lambda|X} : X \longrightarrow \mathbb{P}^r$ is finite & surjective

Note: This number r will be unique. We'll define the dimension of X as r.

Definition: A <u>linear space</u> $\Lambda \subseteq \mathbb{P}^n$ is a closed projective subvariety defined by l.i. linear equations $\Lambda = V_{proj}(L_0, \dots, L_r)$ where are $L_i \subseteq \mathbb{K}[x_0, \dots, x_n]$ are homogeneous of deg 1.

Note: If $\mathbb{P}^n = \mathbb{P}(V)$ then $\Lambda = \mathbb{P}(W)$ for $W \subseteq L$ subspace. ($W = \bigcap_{i=1}^{r} \ker(L_i)$ for $L_i \in V^*$. Thus $\Lambda \simeq \mathbb{P}^{n-r-1}$ since $r+1 = \dim(L_0, \dots, L_r) \leq V^*$.

- We define the linear projection $\pi_\Lambda : \mathbb{P}^n \dashrightarrow \mathbb{P}^r = \mathbb{P}(V/W)$, $p \longmapsto [L_0(p) : \dots : L_r(p)]$

Note: - The map is well-defined outside Λ.

- Writing $V = W \oplus V/W = W \oplus W^\perp$ we set $p \in \Lambda \iff p = [\underbrace{0 : \dots 0}_{r+1} : \underbrace{* : \dots : *}_{n-r}]$

Then $\pi_\Lambda([a_0, \dots, a_n]) = [a_0 : \dots : a_r]$.

Proposition 2: Let $X \subseteq \mathbb{P}^n$ be a projective variety over $\overline{\mathbb{K}} = \mathbb{K}$, disjoint from a linear subspace Λ. Then $\pi_\Lambda : X \longrightarrow \mathbb{P}^r$ is a finite morphism.

From this statement, we recover a projective version of Noether Normalization.

Corollary 4 (Noether Normalization): Any irreducible projective variety X over $\overline{\mathbb{K}}=\mathbb{K}$ admits a finite surjective morphism $X \longrightarrow \mathbb{P}^r$ for some r.

Proof: Fix $X \subseteq \mathbb{P}^n = \mathbb{P}(V)$ irreducible. If $X = \mathbb{P}^n$, set $r=n$ & $X \xrightarrow{id} \mathbb{P}^n$.

Otherwise, pick $\Lambda = \mathbb{P}(W) \subseteq \mathbb{P}^n$ largest linear subspace disjoint from X. (e.g. $\Lambda = [1:0\ldots 0]$ if $X = H_0 \subseteq \mathbb{P}^n$)

- **Claim:** $\pi_\Lambda(X) = \mathbb{P}(V/W) \simeq \mathbb{P}^r$

  Pf/ Any $p=[L] \in \mathbb{P}^r \smallsetminus X$ will produce $W \subsetneq W+L \subseteq V$ & $\mathbb{P}(W+L) \cap X = \emptyset$
  This would contradict the maximality of Λ.

- Since π_Λ is finite by Proposition 2, we are done. $\square$

Proof of Proposition 2: We proceed by induction on $n-r = \dim(W) \in \{1, \ldots, n\}$

We write $X = V_{proj}(F_1, \ldots, F_s)$ where $F_i \in \mathbb{K}[x_0, \ldots, x_n]$ is homogeneous $\forall i$ of degree d_i

- **Base case:** $n-r=1$

  After an automorphism of $\mathbb{P}^n$, we can assume $\Lambda = pt = \{[0, \ldots, 0, 1]\}$

  Basic case: $s=1$:

  Call $d=d_1$. Since $X \cap \Lambda = \emptyset$ we know x_n^d is present in the support of F_1.

  We consider the projection map: $\Lambda_p : X \subseteq \mathbb{P}^n \longrightarrow \mathbb{P}^{n-1}$
  $$[\underline{a}] \longmapsto [a_0, \ldots a_{n-1}]$$

  Now $Y_i := \Lambda_p^{-1}(\underset{\mathbb{P}^{n-1}}{\overset{\cap}{U_i}}) = \{x_i \neq 0\} \cap X \subseteq \mathbb{P}^n$ is an affine variety in $\mathbb{A}^n = U_i \subseteq \mathbb{P}^n$ for $i = 0, \ldots, n-1$

  Furthermore $Y_i = V(F_1^{(i)})$ where $F_1^{(i)}$ is the dehomogenization of F_1.
  and Y_i is irreducible because X is irreducible & $X \not\subseteq H_i$ (Theorem 2 §19.2).

  The corresponding map $\Lambda_p{}^{\#}_{U_i}$ on coordinate rings is the natural projection

  $$\pi: \Lambda_p{}^{\#}_{U_i} : A = \mathbb{K}[x_0, \ldots \hat{x}_i, \ldots x_{n-1}] \longrightarrow \mathbb{K}[x_0, \ldots \hat{x}_i, \ldots, x_n] / \langle F_1^{(i)} \rangle = A[x_n] / \langle F_1^{(i)} \rangle$$
  $$g \longmapsto g \circ \Lambda_{p_1}$$

  Since it corresponds to $Y_i \hookrightarrow \mathbb{A}^n \longrightarrow \mathbb{A}^{n-1}$ $\quad (\frac{a_0}{a_i}, \ldots \frac{a_n}{a_i}) \longmapsto (\frac{a_0}{a_i}, \ldots, \frac{a_{n-1}}{a_i})$

the map on coord. rings is $\pi: A \longrightarrow A[x_n] \longrightarrow A[x_n]/\langle F_1^{(i)}\rangle$

The map π is injective by construction. Since $F_1^{(i)}$ is monic in x_n, we have that π is finite.

General Case: arbitrary s.

Since $p \notin X$ $\exists$ i with $F_i(p) \neq 0$ ie $x_n^{d_i}$ is in the support of F_i.

After reordering, we may assume $i=1$.

As before, we consider the projection map:
$$\begin{array}{rccc} \Lambda_p : & \underset{\mathbb{P}^n}{\overset{}{X}} \subseteq & \longrightarrow & \mathbb{P}^{n-1} \\ & [\underline{a}] & \longmapsto & [a_0, \dots a_{n-1}] \end{array}$$

Then $Y_j = \Lambda_p^{-1}(\underset{\mathbb{P}^{n-1}}{U_j \subseteq}) = \{x_j \neq 0\} \cap X \subseteq \mathbb{P}^n$ is an affine variety in $\mathbb{A}^n$

with $Y_j = V(F_1^{(j)}, \dots, F_r^{(j)})$ $\quad F_1^{(j)} \subseteq \mathbb{K}[x_0, \dots, \hat{x}_j, \dots x_n]$

Again Y_j is irreducible with $\overline{Y_j} = X$ by Theorem 2 §19.2 $(X \not\subseteq H_j)$

- We have $\pi : \Lambda_p^{\#}|_{U_j} : \underset{\underset{A}{\parallel}}{A = \mathbb{K}[x_0, \dots, \hat{x}_j, \dots x_n]} \longrightarrow \mathbb{K}[x_0, \dots, \hat{x}_j, \dots x_n] / \langle F_1^{(j)}, \dots F_r^{(j)} \rangle = A[x_n]/\langle F_1^{(j)}, \dots F_r^{(j)} \rangle$

This map factors:
$$\begin{tikzcd} A \arrow[rr, "\pi"] \arrow[dr, "\alpha"'] & \arrow[d, phantom, "\circlearrowleft"] & A[x_n]/\langle F_1^{(j)}, \dots, F_r^{(j)} \rangle \\ & A[x_n]/\langle F_1^{(j)} \rangle \arrow[ur, "\beta"'] & \end{tikzcd}$$

α is finite & β is surjective, hence finite. Thus, their composition is finite

Inductive Step: $n - r \geq 1$

After applying an automorphism of $\mathbb{P}^n$, we may assume $\Lambda = \{[0 \dots 0, * \dots *]\}$ ie $\Lambda = V(x_0, \dots, x_r)$. By construction Λ is disjoint from X

The projection π_Λ factors as a composition of projections from points in each $\mathbb{P}^j$ disjoint from $[0, \dots, 1]$ in $\mathbb{P}^{j+1}$ (= image of $\Lambda_p : \mathbb{P}^j \longrightarrow \mathbb{P}^{j-1}$, $\underline{a} \longmapsto [a_0 \dots a_{j-1}]$)

$$\pi_\Lambda : \mathbb{P}^n \xrightarrow{\pi_1} \mathbb{P}^{n-1} \xrightarrow{\pi_2} \cdots\cdots \xrightarrow{\pi_{n-r}} \mathbb{P}^r$$

We set $\Lambda_1 = V(x_0,\dots,x_r) \subseteq \mathbb{P}^{n-1}$ & note $\pi_{\Lambda_1} = \pi_2 \circ \cdots \circ \pi_{n-r} : \mathbb{P}^{n-1} \longrightarrow \mathbb{P}^r$

Since $X \cap \Lambda \neq \emptyset$, then $\pi_1(X) \cap \Lambda_1 \neq \emptyset$ & $\pi_1(X) \subseteq \mathbb{P}^{n-1}$ is irred.

- By the base case: $\pi_1|_X : X \longrightarrow \mathbb{P}^{n-1}$ is finite.
- By the inductive hypotheses: $\pi_{\Lambda'} : \pi_1(X) \longrightarrow \mathbb{P}^r$ is finite.

By Lemma 1 $\pi = \pi_{\Lambda'}|_{\pi_1(X)} \circ \pi_1$ is finite. □

__Definition__: Fix $X \subseteq \mathbb{P}^n$ irreducible projective variety. We say the __dimension__ of X is r (& write $\dim X = r$) if $\exists$ finite surjective morphism $X \to \mathbb{P}^r$.

__Next goal__: Ensure this is well-defined & develop Dimension Theory for Varieties.

§3 Proof of Proposition 1 (optional)

__Proposition 1__: Fix a prevariety $X/\overline{\mathbb{K}} = \mathbb{K}$ & $f_1, \dots f_s \in \mathcal{O}_X(X)$ s.t. $\langle f_1, \dots f_s \rangle = \mathcal{O}_X(X)$. If $D_X(f_i) = \{x \in X : f_i(x) \neq 0\}$ is an affine variety for all i, then X is affine.