Lecture XXXI: Dimension Theory I

TODAY: All our examples of topological spaces are either affine varieties or open subsets of affine varieties (=: a quasi-affine variety) over $\overline{\mathbb{K}} = \mathbb{K}$

§1 Topological dimension:

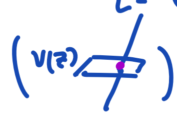
Definition: Fix X a non-empty topological space. The dimension of X is

$$\dim(X) = \sup_{r\geq 0} \{ Z_0 \supsetneq Z_1 \supsetneq \cdots \supsetneq Z_r \ : Z_i \subseteq X \text{ closed \& irreducible} \}$$

Convention: $\dim(\phi) = -1$:

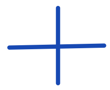
Definition: Fix X a topological space & $Y \subseteq X$ closed & irreducible. The codimension of Y in X is

$$\operatorname{codim}_X(Y) = \sup_{r\geq 0} \{ Z_0 \supsetneq Z_1 \supsetneq \cdots \supsetneq Z_r = Y :: Z_i \subseteq X \text{ closed \& irreducible} \}$$

Examples ① $\dim(\mathbb{A}^1) = 1$, ② $\dim(pt) = 0$, ③ $\dim\Big($  $\Big) = 2$

$L = (x,y)$

$(\mathbb{A}^1 \supsetneq \{pt\})$ $\quad(\{pt\})\quad$ $(\{V(z) \supsetneq V(z,x) \supsetneq V(x,y,z)\})$

④ $\operatorname{codim}_{\mathbb{A}^1}(pt) = 1$, ⑤ $\operatorname{codim}_{V(xy)} V(x) = 0$ , ⑥ $\operatorname{codim}(L) = 0$, ⑦ $\operatorname{codim}(\bar{0}) = 2$

$(\mathbb{A}^1 \supsetneq pt)$ $\quad(V(x))\quad$  (L) $\quad$  $(V(z) \supsetneq V(z,x) \supsetneq pt)$

The motivation for these notions comes from Commutative Algebra:

Definition: Fix a commutative ring R

- The Krull dimension of R is

$$\dim R = \sup_r \{ P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_r \ : P_i \subseteq R \text{ prime } \forall i \}$$

- The codimension or height of a prime P in R is

$$\operatorname{codim}_R P = \max_{r\geq 0} \{ P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_r = P \ : P_i \subseteq R \text{ prime } \forall i \}$$

Remark: If $X \subseteq \mathbb{A}^n$ is an affine variety over $\overline{\mathbb{K}} = \mathbb{K}$, the Nullstellensatz gives

$$Z \text{ is irreducible} \iff I_X(Z) \subseteq \mathbb{K}[X] \text{ is prime}$$

Thus, $\dim X = \dim \mathbb{K}[X]$ & $\operatorname{codim}_X Y = \operatorname{codim}_{\mathbb{K}[X]} I_X(Y)$.

Examples: (1) $\dim(k)=0$ for every field k (only maximal chain: $\{0\}$ because prime ideals are proper)

(2) $\dim(R)=0 \iff$ there are no strict inclusions among prime ideals so all prime ideals are already maximal. More precisely, R is a Noetherian ring, we have $\dim R=0 \iff R$ is Artinian. (this gives Ex 1)

(3) $\dim(\mathbb{K}[t])=1$ because $\mathbb{K}[t]$ is a PID ($0\neq(f)$ is prime $\iff (f)$ is maximal) $\leadsto$ (this gives Ex 0)

Remark: If R is a PID & R is not a field, then $\dim R=1$.
This applies to $R=\mathbb{Z}$ or $\mathbb{K}[t]$.

Main goal: Show $\dim \mathbb{A}^n_{\mathbb{K}}=n$ equiv. $\dim \mathbb{K}[x_1,\dots,x_n]=n$

§2 Properties of dimension:

Next, we show dimension works as expected with respect to basic operations.

Lemma 1: If $Y\subseteq X$ is a topological subspace, then $\dim Y\leq \dim X$

Proof: Fix a sequence of irreducibles in Y: $Z_0\supsetneq Z_1\supsetneq\cdots\supsetneq Z_r$.
Then, taking closures of Z_i in X gives: $\overline{Z_0}\supseteq\overline{Z_1}\supseteq\cdots\supseteq\overline{Z_r}$

Claim 1: $\overline{Z_i}\cap Y=Z_i \;\forall i$ so the inclusions are strict

Claim 2: Z_i irred $\Rightarrow \overline{Z_i}$ is closed & irreducible

Conclude: The chain of closures is a valid chain for $\dim X$, so $r\leq \dim X$.
Taking $\sup_r$ gives $\dim Y\leq\dim X$.

Lemma 2: If X is a topological space & $Y_1,\dots,Y_s\subseteq X$ are closed, then
$$\dim\left(\bigcup_{i=1}^{s}Y_i\right)=\max_{1\leq i\leq s}\dim(Y_i)$$
(this gives Ex 3)

This applies if X is Noetherian & $Y_1,\dots Y_s$ are the irreducible components of X.

Proof: Since Y_i is closed in X we can replace X by $Y:=\bigcup_{i=1}^{s}Y_i$.

- $Y_i\subseteq Y$ subspace $\underset{\text{Lemma 1}}{\Longrightarrow}$ $\dim(Y_i)\leq\dim(Y)\;\forall i$
- Given $Z_0\supsetneq Z_1\supsetneq\cdots\supsetneq Z_r$ sequence of closed irreducibles in Y, then $\exists i$ with $Z_0\subseteq Y_i$

Thus so $r\leq\max_{1\leq i\leq s}\dim(Y_i)$ & by taking $\sup_r$ we get $\dim Y\leq\max_{1\leq i\leq s}\{\dim Y_i\}$.

• Our next result reduces the computation of dimensions of quasi-affine varieties to that of affine varieties.

Lemma 3: Fix X a topological space & $X = U_1 \cup \cdots \cup U_s$ with $U_i \subseteq X$ open $\forall i$.
Then: $\dim(X) = \max_{1 \leq i \leq s} \dim(U_i)$

Proof: • $\geq$ follows by Lemma 1 ($U_i \subseteq X$ is a topological subspace)
• For $\leq$: fix $Z_0 \supsetneq Z_1 \supsetneq \cdots \supsetneq Z_r$ chain of closed irreducibles in X
Pick i such that $Z_r \cap U_i \neq \emptyset \implies Z_j \cap U_i \neq \emptyset \quad \forall j = 0, \ldots, r$
__Claim 1:__ $Z_j \cap U_i \subseteq Z_j$ is open & Z_j is irreducible, so $\overline{Z_j \cap U_i} = Z_j$
Thus: $Z_0 \cap U_i \supsetneq Z_1 \cap U_i \supsetneq \cdots \supsetneq Z_r \cap U_i$
is a chain of closed irreducibles of U_i (inclusions are strict by Claim 1, $Z_j \cap U_i \subseteq U_i$ is irred)
Thus $\dim U_i \geq r$, so $\max_{1 \leq i \leq s} \dim(U_i) \geq \sup_r = \dim(X)$ □

Lemma 4: Fix X topological space, $Y \subseteq X$ closed irreducible & $U \subseteq X$ open with $U \cap Y \neq \emptyset$. Then $\operatorname{codim}_U(U \cap Y) = \operatorname{codim}_X(Y)$.

Proof: Exercise.

Lemma 5: Fix X Noetherian topological space with irreducible decomposition $X = X_1 \cup \cdots \cup X_s$. Assume $Y \subseteq X$ is closed & irreducible in X. Then:

$$\operatorname{codim}_X Y = \max_{1 \leq i \leq s} \{ \operatorname{codim}_{X_i}(Y) \mid Y \subseteq X_i \}$$

(this gives Ex 5, 6, 7)

Proof: An irreducible closed subset of X always lies in some irred component of X.
__($\leq$):__ Any chain of closed irreducibles $Z_0 \supsetneq Z_1 \supsetneq \cdots \supsetneq Z_r = Y$ will satisfy $Z_0 = X_i$ for some i. Thus $r \leq \operatorname{codim}_{X_i} Y$. This gives ($\leq$).
__($\geq$)__ Conversely, any chain of closed irreducibles in X_i ending at Y is a chain in X. So ($\geq$) is clear.

Remark: For Ex 3, 6 & 7 we need to know $\dim \mathbb{A}^2 = 2$. Once this is assumed, we get the results from Lemmas 2 & 5.