# Lecture XXXII: Dimension Theory II

**Recall:** ① $X$ topological space

- $\dim(X) = \sup_r \{ Z_0 \supsetneq Z_1 \supsetneq \cdots \supsetneq Z_r : Z_i \subseteq X \text{ closed an irreducible} \}$
- For $Y \subseteq X$ closed & irreducible: $\operatorname{codim}_X(Y) = \sup_Z \{ Z_0 \supsetneq Z_1 \supsetneq \cdots \supsetneq Z_r = Y : Z_i \subseteq X$ $Z_i$ closed irreducible $\forall i \}$

② $R$ commutative ring $\dim R = \sup_r \{ \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subseteq \mathfrak{p}_r : \mathfrak{p}_i \subset R \text{ prime } \forall i \}$

If $\mathfrak{p} \subseteq R$ prime $\operatorname{codim}_R \mathfrak{p} = \sup \{ \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_r = \mathfrak{p} : \mathfrak{p}_i \subseteq R \text{ prime } \forall i \}$

**Remark:** If $X$ affine variety $Y \subseteq X$ closed, irred (1) $\dim X = \dim \mathbb{K}[X]$

(2) $\operatorname{codim}_X Y = \operatorname{codim}_{\mathbb{K}[X]} I_Y(X)$

**Lemma:** Fix $X$ Noetherian topological space & $X = X_1 \cup \cdots \cup X_s$ irreducible decomp.

- $\dim(X) = \max_{1 \leq i \leq s} \dim(X_i)$
- If $Y \subseteq X$ closed irred $\operatorname{codim}_Y(X) = \max_{1 \leq i \leq s} \{ \operatorname{codim}_{X_i}(Y) : Y \subseteq X_i \}$

**Lemma:**
- If $Y \subseteq X$ is a topological subspace, then $\dim Y \leq \dim X$
- If $U \subseteq X$ open & $Y \subseteq X$ closed irred with $U \cap Y = \emptyset$ $\operatorname{codim}_X Y = \operatorname{codim}_U U \cap Y$.

## §1 Finite morphisms and dimension theory.

Finite morphisms interact nicely with dimension

**Theorem 1:** Fix $X, Y$ affine varieties over $\overline{\mathbb{K}} = \mathbb{K}$ & $\varphi : X \longrightarrow Y$ finite surjective morphism. Then, $\dim X = \dim Y$.

Moreover, if $Z \subseteq X$ is closed & irreducible, we have $\operatorname{codim}_X Z = \operatorname{codim}_Y \varphi(Z)$.

**Proof:** We will use that finite morphisms are closed (see Theorem 1 §29.1)

If $Z_0 \supsetneq Z_1 \supsetneq \cdots \supsetneq Z_r$ is a sequence of closed irred sets in $X$, then

$$\varphi(Z_0) \supseteq \varphi(Z_1) \supseteq \cdots \supseteq \varphi(Z_r)$$

is a sequence of closed sets in $Y$.

**Claim 1:** $Z_i$ irred, then $\overline{\varphi(Z_i)} = \varphi(Z_i)$ is irreducible

**Claim 2:** $Z_k \subsetneq Z_\ell \Rightarrow \varphi(Z_k) \subsetneq \varphi(Z_\ell)$ by Lemma 1(1)

**Conclusion:** $\dim X \leq \dim Y$.

• For the other inclusion: suppose we are given a sequence $W_0 \supsetneq W_1 \supsetneq \cdots \supsetneq W_s$ of closed irred in $Y$. We want to build a chain

$$T_0 \supsetneq T_1 \supsetneq T_2 \supsetneq \cdots \supsetneq T_s \quad \text{in } X$$

where
- $T_i$ closed & irred $\forall i$
- $\psi(T_i) = W_i$ (so inclusions are proper)

We use Lemma 6 below to achieve this. Indeed, by Lemma 1.(2) we can find $T_0 \subseteq X$ closed & irreducible with $\psi(T_0) = W_0$. By Lemma 1.(3) $\exists$ $T_1 \subseteq T_0$ closed & irreducible in $X$ with $\psi(T_1) = W_1$. Repeating this argument, we get a chain

$$T_0 \supsetneq T_1 \supsetneq \cdots \supsetneq T_r \quad \text{with } T_i \text{ closed \& irreducible in } X \ \forall i.$$

<u>Conclusion</u>: $\dim Y = \sup_r \leq \dim X$.

• The proof of the 2nd assertion is analogous, but now all our chains must end at $Z$ or $\psi(Z)$ respectively. For this we use Lemma 1 (2) replacing $X$ & $Y$ with □ $Z$ & $\psi(Z)$, respectively. Since $\psi$ is finite $\psi|_Z$

Next, we write the statement of the technical results on finite maps that we need. The proof will follow from analogous results for finite ring homomorphisms.

<u>Lemma 1</u>: Fix $X \xrightarrow{\psi} Y$ finite morphism of affine varieties over $\overline{K} = K$.

Then (1) If $Z_1 \subseteq Z_2$ are irreducible closed subsets of $X$, then $\psi(Z_1) \subsetneq \psi(Z_2)$ are irreducible closed subsets of $Y$

(2) If $\psi$ is surjective, then given any irreducible closed subset $W$ of $Y$, there exists $Z \subseteq X$ closed & irreducible with $\psi(Z) = W$

(3) If $Z_1 \subseteq X$ is closed & irreducible & $W_1 \supseteq W_2$ are irreducible, closed subsets of $Y$, with $W_1 = \psi(Z_1)$, then there exists $Z_2 \subseteq Z_1$ irreducible & closed in $X$ with $W_2 = \psi(Z_2)$.

Pf/ We may assume $\psi$ is dominant, by replacing $Y$ with $\overline{\psi(X)} = \psi(X)$ (recall finite morphisms are closed) Write $\varphi = \psi^{\#}_Y : \mathcal{O}_Y(Y) \longrightarrow \mathcal{O}_X(X)$

Recall that $\varphi = \psi^{\#}_Y : K[Y] \hookrightarrow K[X]$ is injective because $\psi$ is dominant.

Claim: Write $Z = V_x(q)$ for $q \subseteq \mathbb{K}[X]$ prime ideal. Then

$$\psi(Z_j) \underset{\psi \text{ closed}}{=} \overline{\psi(Z)} = V(\varphi^{-1}(q))$$

(1) We set $Z_1 = V(q_1)$ $Z_2 = V(q_2)$ with $q_1, q_2 \subseteq \mathbb{K}[X]$ prime, $q_1 \subsetneq q_2$ by the Nullstellensatz.

Lemma 2 (1) below says $q_1 \subsetneq q_2$ prime $\Rightarrow$ $\mathfrak{p}_1 := \varphi^{-1}(q_1) \subsetneq \varphi^{-1}(q_2) =: \mathfrak{p}_2$ are also prime.

So by the Nullstellensatz & the claim we have:

$$\psi(Z_1) := V(\varphi^{-1}(q_1)) \supsetneq V(\varphi^{-1}(q_2)) =: \psi(Z_2)$$

(2) This follows from Lemma 2 (2) below. Set $W = V(\mathfrak{p})$ $\mathfrak{p} \in \mathbb{K}[Y]$ & take $Z = V(q\ \mathbb{K}[X])$ where $\varphi^{-1}(q) = \mathfrak{p}$

(3) This follows from Lemma 2 (3) below. Set $W_1 = V(\mathfrak{p}_1)$, $W_2 = V(\mathfrak{p}_2)$ with $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$ prime. & set $q_1 \subseteq \mathbb{K}[X]$ prime with $\varphi^{-1}(q_1) = \mathfrak{p}_1$. Then pick $q_2 \supseteq q_1$ prime in $\mathbb{K}[X]$ with $\varphi^{-1}(q_2) = \mathfrak{p}_2$ $\square$

The analogous result for finite morphisms of rings is:

**Lemma 2:** Fix $\varphi : A \longrightarrow B$ finite morphism of rings. Then:

(0) Fix $q \subseteq B$ prime ideal & set $\mathfrak{p} := \varphi^{-1}(q)$. Then: $q \subseteq B$ is maxl $\Leftrightarrow$ $\mathfrak{p} \subseteq A$ is max.

(1) [Incomparability] If $q_1 \subsetneq q_2$ are prime ideals of $B$, then $\varphi^{-1}(q_1) \subsetneq \varphi^{-1}(q_2)$

(2) [Lying Over] If $\varphi$ is injective, then $\forall\ \mathfrak{p} \subseteq A$ prime ideal, we can find $q \subseteq B$ prime with $\varphi^{-1}(q) = \mathfrak{p}$.

(3) [Going-Up] Given $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$ prime ideals in $A$ & a prime $q_1 \subseteq B$ with $\varphi^{-1}(q_1) = \mathfrak{p}_1$ $\exists$ prime ideal $q_2 \subseteq B$ with $q_1 \subseteq q_2$ & $\varphi^{-1}(q_2) = \mathfrak{p}_2$.

Pictorially:

$$\begin{array}{ccccc} & q_1 & \subseteq & \boxed{q_2} & \text{in } B \\ & | & & | & \\ \varphi^{-1}(q_1) = & \mathfrak{p}_1 & \subseteq & \mathfrak{p}_2 = \varphi^{-1}(q_2) & \text{in } A \end{array}$$

Proof: (0) By construction we have an induced map $\overline{\varphi} : A/\mathfrak{p} \hookrightarrow B/\mathfrak{q}$

Furthermore, $\overline{\varphi}$ is a finite & injective map ($\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$) between domains.

Claim: $A' = A/\mathfrak{p}$ is a field $\iff$ $B' = B/\mathfrak{q}$ is a field.

Pf/ ($\Leftarrow$) Fix $u \in A' \setminus \{0\}$ & let $b = \frac{1}{\varphi(u)} \in B'$. Since $B' | A'$ is integral

$\exists\, n \in \mathbb{N}$ & $a_1, \ldots, a_n \in A'$ with $b^n + \varphi(a_1) b^{n-1} + \cdots + \varphi(a_n) = 0$ in $B'$

then $\left(\frac{1}{\varphi(u)}\right)^n + \varphi(a_1)\left(\frac{1}{\varphi(u)}\right)^{n-1} + \cdots + \varphi(a_n) = 0$ yields

$$\frac{1}{\varphi(u)} + \varphi(a_1) + \varphi(a_2)\varphi(u) + \cdots + \varphi(a_n)\varphi(u)^{n-1} = 0$$

ie $\frac{1}{\varphi(u)} = -\varphi(a_1 + a_2 u + \cdots + a_n u^{n-1}) \in \varphi(A)$

Then $\frac{1}{u} = a_1 + a_2 u + \cdots + a_n u^{n-1}$ in $A$ by the injectivity of $\varphi$.

($\Rightarrow$) Assume $A'$ is a field & pick $b \in B' \setminus \{0\}$. Since $B' | A'$ is integral, then $\exists\, n \in \mathbb{N}$ (minimal) & $a_1, \ldots, a_n \in A'$ with $b^n + \varphi(a_1) b^{n-1} + \cdots + \varphi(a_n) = 0$ in $B'$

By minimality & $b \neq 0$ we have $\varphi(a_n) \neq 0$ ie $a_n \neq 0$.

We write $0 = b^n + \varphi(a_1) b^{n-1} + \cdots + \varphi(a_n) = b(b^{n-1} + \varphi(a_1) b^{n-2} + \cdots + \varphi(a_1)) + \varphi(a_n)$

equivalently: $1 = b\,\underbrace{\left(-\varphi\left(\tfrac{1}{a_n}\right)(b^{n-1} + \varphi(a_1) b^{n-2} + \cdots + \varphi(a_n)\right)}_{\text{in } B}$ in $B$

Thus, $b \in B$ is invertible. $\square$

(1) Note: $\{\mathfrak{q} \subseteq B$ prime with $\varphi^{-1}(\mathfrak{q}) \subseteq \mathfrak{p}\} \xleftrightarrow{\text{1-to-1}} \{\tilde{\mathfrak{q}} \in B_{\mathfrak{p}} = A_{\mathfrak{p}} B$ prime with $\mathfrak{p} B_{\mathfrak{p}} \subseteq \tilde{\mathfrak{q}}\}$

$$\mathfrak{q} \longmapsto \mathfrak{q} B_{\mathfrak{p}} / \mathfrak{p} B_{\mathfrak{p}}$$

(Localization is exact)

- Since $\varphi$ is finite the induced morphism $\varphi_{\mathfrak{p}} : A_{\mathfrak{p}} \longrightarrow B_{\mathfrak{p}}$ is finite & so is $\overline{\varphi}_{\mathfrak{p}} : \underbrace{A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}}_{\text{field}} \longrightarrow B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}} = B \otimes_A A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$

Given $\mathfrak{q}_1$ & $\mathfrak{q}_2$ as in (1), assume $\varphi^{-1}(\mathfrak{q}_1) = \mathfrak{p} = \varphi^{-1}(\mathfrak{q}_2)$ By (0) we have that $\mathfrak{q}_1 B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}$ & $\mathfrak{q}_2 B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}$ are maximal ideals of $B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}$

However $q_1 B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p} \subsetneq q_2 B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$ by Note above, so this cannot happen!

(2) By construction, we know $B$ is a f.g. $A$-module, so $B_\mathfrak{p}$ is a f.g. $A_\mathfrak{p}$ mod
$0 \neq A_\mathfrak{p} \hookrightarrow A_\mathfrak{p} B =: B_\mathfrak{p}$ so $B_\mathfrak{p} \neq 0$.
By Nakayama's Lemma (Lecture §29.1) $B_\mathfrak{p} \neq \mathfrak{p} B_\mathfrak{p}$
Now $B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p} \neq 0$ so it contains a prime ideal $\tilde{q} \in B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$.
By the note: $\tilde{q} = q B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$ for some $q \subseteq B$ prime ideal with $\varphi^{-1}(q) = \mathfrak{p}$.

(3) Fix $\mathfrak{p}_1, \mathfrak{p}_2, q_1$ as in the statement & consider the induced morphism of rings $\overline{\varphi}: A/\mathfrak{p}_1 \longrightarrow B/q_1$

- Since $\varphi(q_1) = \mathfrak{p}_1$, $\overline{\varphi}$ is injective
- $\overline{\varphi}$ is finite because $\varphi$ is finite

Applying (2) $\exists\, \tilde{q}_2 \subseteq B/q_1$ prime with $\overline{\varphi}(\tilde{q}_2) = \mathfrak{p}_2/\mathfrak{p}_1$
By construction $\tilde{q}_2 = q_2/q_1$ with $q_2$ prime in $B$ & $\varphi^{-1}(q_2) = \mathfrak{p}_2$. $\square$

## §2 Properties of Krull dimension:

⚠ Noetherianness of a ring and finite dimension are not related, although it's hard to find examples where only one of these properties fails.

Example 1 (Nagata) $A = \mathbb{K}[x_n : n \in \mathbb{N}]$ & take an increasing sequence $m_1, m_2, \dots$ with $m_{i+1} - m_i > m_i - m_{i-1}$. $\forall i$ Set $m_0 = 0$.
Take $\mathfrak{p}_i = (x_{m_i+1}, \dots, x_{m_{i+1}})$ & set $S = A \setminus \bigcup_{i=0}^{\infty} \mathfrak{p}_i$

(1) $S$ is multiplicatively closed
(2) $R = S^{-1}A$ is Noetherian
(3) $S^{-1}\mathfrak{p}_i \subseteq R$ has codimension $m_{i+1} - m_i \xrightarrow[i\to\infty]{} \infty$

Conclude: $R$ is Noetherian & $\dim R = \infty$

Example 2 $R = \mathbb{K}[x_n : n \in \mathbb{N}]/(x_n^2 : n \in \mathbb{N})$ is not Noetherian, but $\dim R = 0$

## Lemma 3: For every $\mathfrak{p} \subseteq R$ prime we have:

(1) $\operatorname{codim}_R \mathfrak{p} = \dim R_{\mathfrak{p}}$  (2) $\dim R/\mathfrak{p} + \operatorname{codim}_R \mathfrak{p} \leq \dim R$

**Proof:** (1) $\tilde{\mathfrak{q}} \subseteq R_{\mathfrak{p}} \longleftrightarrow \mathfrak{q} = \tilde{\mathfrak{q}} \cap R \subseteq R$ prime ideal containing $\mathfrak{p}$

(2) Any pair of chains $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_r = \mathfrak{p}$ of primes in $R$

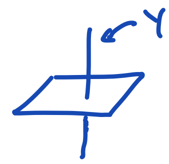
$\underset{\uparrow \, R/\mathfrak{p} \text{ domain}}{\{0\}} = \mathfrak{q}_0 \subseteq \cdots \subseteq \mathfrak{q}_s$ ——— $R/\mathfrak{p}$

produce a chain $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_r = \mathfrak{p} = \pi^{-1}(\mathfrak{q}_0) \subsetneq \pi^{-1}(\mathfrak{q}_1) \subsetneq \cdots \subseteq \pi^{-1}(\mathfrak{q}_s)$ of prime ideals of $R$ of length $r+1+s = r+s+1$ where $\pi : R \longrightarrow R/\mathfrak{p}$

Taking $\sup_{r,s}$ gives $\underbrace{\operatorname{codim}_R \mathfrak{p}}_{= \sup_r} + \underbrace{\dim R/\mathfrak{p}}_{= \sup_s} \leq \dim R$.

**Corollary 1:** Given $X$ affine variety & $Y \subseteq X$ irreducible affine variety, we have

$$\dim X \geq \dim Y + \operatorname{codim}_X Y.$$

**Remarks:** ① The equality need not hold in examples $X =$ 

$\dim X = 2$
$\dim Y = 1$
$\operatorname{codim}_X Y = 0$

② However, if $R = \mathbb{K}[x]$ has the property that every maximal chain of prime ideals has the same dimension, then we would have $=$ in the Corollary. This is precisely what happens for $X = \mathbb{A}^n$!