Lecture XXXIII: Dimension Theory III

GOAL: Show $\dim \mathbb{K}[x_1, \ldots, x_n] = n$ for any field $\mathbb{K}$

We will need 2 lemmas from Lecture 32.

Lemma 1: Fix $\varphi : A \longrightarrow B$ finite morphism of rings. Then:

(1) [Incomparability] If $q_1 \subsetneq q_2$ are prime ideals of B, then $\varphi^{-1}(q_1) \subsetneq \varphi^{-1}(q_2)$

(2) [Lying Over] If φ is injective, then $\forall\, \mathfrak{P} \subseteq A$ prime ideal, we can find $q \subseteq B$ prime with $\varphi^{-1}(q) = \mathfrak{P}$.

(3) [Going-Up] Given $\mathfrak{P}_1 \subseteq \mathfrak{P}_2$ prime ideals in A & a prime $q_1 \subseteq B$ with $\varphi^{-1}(q_1) = \mathfrak{P}_1$ $\exists$ prime ideal $q_2 \subseteq B$ with $q_1 \subseteq q_2$ & $\varphi^{-1}(q_2) = \mathfrak{P}_2$.

Pictorially:

$$\begin{array}{ccccl} & q_1 & \subseteq & \boxed{q_2} & \quad \text{in } B \\ & | & & | & \\ \varphi^{-1}(q_1) = & \mathfrak{P}_1 & \subseteq & \mathfrak{P}_2 = \varphi^{-1}(q_2) & \quad \text{in } A \end{array}$$

Lemma 2: For any $\mathfrak{P} \subseteq R$ prime we have:

(1) $\operatorname{codim}_R \mathfrak{P} = \dim R_{\mathfrak{P}}$ (2) $\dim R/\mathfrak{P} + \operatorname{codim}_R \mathfrak{P} \leq \dim R$

Remark: Equality in (2) can be strict!

§1 Krull dimension & finite extensions:

Q: How does dimension behave with respect to finite extensions?

Intuition: Finite extensions of quotients of polynomial rings correspond to finite dominant maps, so dimension in these cases should be invariant under integral extensions (by Theorem 1 §32.1).

Lemma 3: For any finite ring extension $R \subseteq R'$ we have $\dim R' = \dim R$

Proof: We show the double inequality.

($\geq$) Fix $\mathfrak{P}_0 \subsetneq \mathfrak{P}_1 \subsetneq \cdots \subsetneq \mathfrak{P}_r$ chain of prime ideals in R. Using Lying-over & going up (Lemma 2 (2)&(3)) on $\varphi = \operatorname{inc} : R \hookrightarrow R'$ we can find a chain of maximal ideals in R' $\quad \mathfrak{q}_0 \subsetneq \mathfrak{q}_1 \subsetneq \cdots \subsetneq \mathfrak{q}_r$ with $\varphi^{-1}(\mathfrak{q}_i) = \mathfrak{P}_i \quad \forall i$

Then $r \leq \dim R'$. Taking $\sup_r$ gives $\dim R = \sup_r \leq \dim R'$.

($\leq$) Given a chain $\mathfrak{P}_0' \subsetneq \mathfrak{P}_1' \subsetneq \cdots \subsetneq \mathfrak{P}_r'$ of prime ideals in R'. We get a chain $\mathfrak{P}_0 \subseteq \mathfrak{P}_1 \subseteq \cdots \subseteq \mathfrak{P}_r$ of prime ideals in R via $\mathfrak{P}_i = \mathfrak{P}_i' \cap R = \varphi^{-1}(\mathfrak{P}_i')$

By Incomparability (Lemma 2 (1)), $\mathfrak{P}_i \neq \mathfrak{P}_{i+1}$ because $\mathfrak{P}_i' \subsetneq \mathfrak{P}_{i+1}'$. So, the chain in R is strict and thus $r \leq \dim R$. Taking $\sup_r$ gives $\dim R' = \sup_r \leq \dim R$. $\square$

§2 Dimension of $\mathbb{A}^n_{\mathbb{K}}$ is n:

Theorem 1 (Dimension of Polynomial Rings)

Fix $\mathbb{K}$ any field (not necessarily algebraically closed) & $n \in \mathbb{N}$. Then

(1) $\dim \mathbb{K}[x_1, \ldots, x_n] = n$

(2) All maximal chains (w.r.t. inclusion) of prime ideals in $\mathbb{K}[x_1, \ldots, x_n]$ have length n.

Corollary 1: Any affine variety $X \subseteq \mathbb{A}^n_{\mathbb{K}}$ has dimension at most n. Equality holds if & only if $X = \mathbb{A}^n_{\mathbb{K}}$

Proof: Write $X = X_1 \cup \cdots \cup X_s$ irred. decomposition of X. Then by Lemma 2 §31.2 we have $\dim X = \max_{1 \leq i \leq s} \dim X_i$. Now $\mathbb{K}[X_i] = \dfrac{\mathbb{K}[x_1, \ldots x_n]}{I(X_i)}$ & $I(X_i) = \mathfrak{P}_i$ is a prime ideal in $R = \mathbb{K}[x_1, \ldots, x_n]$.

Since $\mathfrak{P}_i$ is always part of a maximal chain of prime ideals in X by Thm 2 we get:

$$\dim X_i = \dim \mathbb{K}[X_i] = n - \operatorname{codim}_R \mathfrak{P}_i \leq n$$

and equality holds if and only if $\operatorname{codim}_R \mathfrak{P}_i = 0$, ie $\mathfrak{P}_i = 0$. $\square$

Proof of Theorem 1: We prove both statements by induction on n ((2) $\Rightarrow$ (1) so only need to show (2)) Write $R_n := \mathbb{K}[x_1, \ldots x_n]$

Base cases: $n=0$ is tautological; $n=1$ is true since $\mathbb{K}[t]$ is a PID (§31.1)

Inductive Step: $n \geq 1$ Fix $\mathfrak{P}_0 \subsetneq \cdots \subsetneq \mathfrak{P}_m$ a chain of prime ideals in $\mathbb{K}[x_1, \ldots x_n]$

We need to show: (1) $m \leq n$ & (2) equality holds if the chain is maximal

If the chain is maximal, we know that:

- $\mathfrak{P}_0 = 0$
- $\mathfrak{P}_1$ is a minimal nonzero prime ideal $\quad (\Longrightarrow \mathfrak{P}_1 = (f)$ for some $f \neq 0)$
  - ↳ pick $f \in \mathfrak{P}_1 \setminus \{0\}$ of minimal total degree ($\Rightarrow$ it's irreducible because $\mathfrak{P}_1$ is prime)
  - $\mathfrak{P}_1 = (f)$ because R_n is a UFD
- $\mathfrak{P}_m$ is a maximal ideal

By Noether Normalization, we can consider $\mathbb{K}[u_1,\dots,u_{\underline{n}}] \xhookrightarrow{\varphi} \mathbb{K}[x_1,\dots,x_n]$ integral where $\tilde{f} = f(x_1(\underline{u}),\dots,x_n(\underline{u}))$ is monic in u_n. $\quad (\underline{r} = \operatorname{tr\,deg}_{\mathbb{K}} \mathbb{K}[x_1,\dots,x_n] = n)$

Thus, ① $\tilde{f} \in \mathbb{K}[u_1,\dots,u_n]$, $\bar{u}_n \in \mathbb{K}[u_1,\dots,u_n]/(\tilde{f})$ is integral over $\mathbb{K}[u_1,\dots,u_{n-1}]$

② $\mathbb{K}[u_1,\dots,u_n]/(\tilde{f})$ is integral over $\mathbb{K}[u_1,\dots,u_{n-1}]$, so finite. $\quad \tilde{\mathfrak{P}}_1 = \varphi^{-1}(f) = (\tilde{f})$.

This gives a way to transfer chains on $R_n^{(\underline{u})}$ to chains in $R_{n-1}^{(\underline{u})}$ via those in $\mathbb{K}[u_1,\dots,u_n]/(\tilde{f})$

$$\begin{array}{llllll}
\mathbb{K}[x_1,\dots,x_n]: & (0)=\mathfrak{P}_0 & \subsetneq \mathfrak{P}_1 & \subsetneq \cdots & \subsetneq \mathfrak{P}_m & \\
\varphi \uparrow \text{ finite} & & & & & \\
\mathbb{K}[u_1,\dots,u_n]: & \tilde{\mathfrak{P}}_0 = \varphi^{-1}(\mathfrak{P}_0) & \subsetneq \tilde{\mathfrak{P}}_1 = \varphi^{-1}(\mathfrak{P}_1) & \subsetneq \cdots & \subsetneq \tilde{\mathfrak{P}}_m = \varphi^{-1}(\mathfrak{P}_m) & \text{(Incomparability)} \\
\mathbb{K}[u_1,\dots,u_n]/\tilde{\mathfrak{P}}_1 & & (0) = \tilde{\mathfrak{P}}_1/\tilde{\mathfrak{P}}_1 & \subsetneq \cdots & \subsetneq \tilde{\mathfrak{P}}_m/\tilde{\mathfrak{P}}_1 & \text{(contraction)} \\
\tilde{\varphi} \uparrow \text{ finite} & & & & & \\
\mathbb{K}[u_1,\dots,u_{n-1}] & & (\tilde{\mathfrak{P}}_1/\tilde{\mathfrak{P}}_1) \cap \mathbb{K}[u_1,\dots,u_{n-1}] & \subsetneq \cdots & \subsetneq (\tilde{\mathfrak{P}}_m/\tilde{\mathfrak{P}}_1) \cap \mathbb{K}[u_1,\dots,u_{n-1}] & \text{(Incomparability)}
\end{array}$$

In all rows, the chains are strict by Incomparability (Lemma 2(1))

<u>Claim</u>: These 4 chains are also maximal

Pf/ • The chain in $\mathbb{K}[u_1,\dots,u_n]$ is maximal because otherwise by Lying over & Going Up we will produce a chain extending the top one. Indeed,

$\tilde{\mathfrak{P}}_i \subseteq \mathfrak{P} \subseteq \tilde{\mathfrak{P}}_{i+1}$ will give $\tilde{\mathfrak{P}}_i = \varphi^{-1}(\mathfrak{P}_i)$ & $\tilde{\mathfrak{P}}_{i+1} = \varphi^{-1}(\mathfrak{P}_{i+1})$

$$\begin{array}{ccccccl}
\mathfrak{P}_i & \subseteq & \boxed{\mathfrak{q}} & \subseteq & \mathfrak{P}_{i+1} & \text{in} & R = \mathbb{K}[x_1,\dots,x_n] \\
| & & | & & | & & \quad \uparrow \varphi \\
\tilde{\mathfrak{P}}_i & \subsetneq & \mathfrak{P} & \subsetneq & \tilde{\mathfrak{P}}_{i+1} & \text{in} & R' = \mathbb{K}[u_1,\dots,u_n]
\end{array}$$

Key: By working with $(R')_{\tilde{\mathfrak{P}}_{i+1}} \xrightarrow{\varphi_{\tilde{\mathfrak{P}}_{i+1}}} (R)_{\mathfrak{P}_{i+1}}$ (which is again finite since $\tilde{\mathfrak{P}}_{i+1} = \varphi^{-1}(\mathfrak{P}_{i+1})$ & localization is exact), we can find $\mathfrak{q}' \subseteq (R)_{\mathfrak{P}_{i+1}}$ prime with $\varphi^{-1}_{\tilde{\mathfrak{P}}_{i+1}}(\mathfrak{q}') = \mathfrak{P}$ & $\mathfrak{P}_i R_{\mathfrak{P}_{i+1}} \subseteq \mathfrak{q}' \subseteq \mathfrak{P}_{i+1} R_{\mathfrak{P}_{i+1}}$

Thus $\mathfrak{q} := \mathfrak{q}' \cap R$ is a prime ideal with $\mathfrak{P}_i \subsetneq \mathfrak{q} \subsetneq \mathfrak{P}_{i+1}$. This cannot happen because our chain in row 1 was already maximal.

Conclusion: The chain in row 2 is maximal.

- Taking quotient preserves maximality. So the chain in row 3 is also maximal.
- From row 4 we use the same methods as for those saying "row 1 maximal $\Rightarrow$ row 2 is also maximal", to conclude that row 4 is maximal because the one in row 3 was. $\square$

By inductive hypothesis; looking at the maximal chain in row 4 we get:
- $m-1 \leq n-1$.
- any maxl chain in $\mathbb{K}[u_1, \ldots u_{n-1}]$ has the same length $= n-1$

Therefore, $m \leq n$ & any maximal chain in $\mathbb{K}[x_1, \ldots x_n]$ has length n $\square$

- We end this section with an alternative definition of dimension of irreducible varieties in terms of transcendence degrees of function fields over $\mathbb{K}$.

Recall the following statement for Lecture 19

Theorem 1 §19.1: If X, Y are affine varieties over $\overline{\mathbb{K}} = \mathbb{K}$ & $\varphi : X \longrightarrow Y$ is a finite surjective regular map then $\dim X = \dim Y$.

Corollary 2: If X is an irreducible affine variety over $\overline{\mathbb{K}} = \mathbb{K}$, then $\dim X = \operatorname{trdeg}_{\mathbb{K}} \mathbb{K}(X)$

In particular, $\dim X < \infty$ & for every $U \subseteq X$ non-empty open, $\dim U = \dim X$.

Proof: By Noether Normalization we can construct a finite dominant (hence surjective) map $X \longrightarrow \mathbb{A}^r$ where $r = \operatorname{trdeg}_{\mathbb{K}} \mathbb{K}(X)$. By Theorem 1 §19.1, $\dim(X) = \dim(\mathbb{A}^r_{\mathbb{K}}) = r$.

- The statement for $\dim X < \infty$ follows from the fact that $\operatorname{trdeg}_{\mathbb{K}} \mathbb{K}(X) < n$ if $X \subseteq \mathbb{A}^n_{\mathbb{K}}$
- The claim about $U \subseteq X$ is a consequence of the fact that $\mathbb{K}(U) = \mathbb{K}(X)$. $\square$

§3 Krull's principal ideal Theorem:

Theorem 2 (Krull's Principal Ideal Theorem) Let R be Noetherian ring & let $a \in R$. Then, every minimal prime ideal $\mathfrak{P}$ over (a) satisfies $\operatorname{codim}_R \mathfrak{P} \leq 1$.

In order to prove Theorem 2 we make use of symbolic powers of prime ideals:

These ideals are used to study singularities of varieties defined over fields with positive characteristic.

<u>Definition:</u> Let R be Noetherian ring & $\mathfrak{p} \subseteq R$ a prime ideal. For each $n \in \mathbb{N}$ we define the <u>n^{th} symbolic power of $\mathfrak{p}$</u> as:

$$\mathfrak{p}^{(n)} = \{ a \in R : ba \in \mathfrak{p}^n \text{ for some } b \in R \setminus \mathfrak{p} \}$$

<u>Lemma 4:</u> Given R, $\mathfrak{p}$ & n as above, we have:

(1) $\mathfrak{p}^{(n)} \subseteq R$ is an ideal & $\mathfrak{p}^{(n+1)} \subseteq \mathfrak{p}^{(n)}$

(2) $\mathfrak{p}^n \subseteq \mathfrak{p}^{(n)} \subseteq \mathfrak{p}$

(3) $\mathfrak{p}^{(n)}$ is $\mathfrak{p}$-primary

(4) $\mathfrak{p}^{(n)} R_{\mathfrak{p}} = \mathfrak{p}^n R_{\mathfrak{p}}$

<u>Proof:</u> (1) We show that $\mathfrak{p}^{(n)}$ is an ideal by checking 3 properties:

- $0 = 1 \cdot 0^n \in \mathfrak{p}^n$ and $1 \notin \mathfrak{p}$.
- $a_1, a_2 \in \mathfrak{p}^{(n)}$ then $\exists\, b_1, b_2 \notin \mathfrak{p}$ such that $b_1 a_1 \in \mathfrak{p}^n$ & $b_2 a_2 \in \mathfrak{p}^n$.
  Then $(b_1 b_2) a_1 \in \mathfrak{p}^n$ & $(b_1 b_2) a_2 \in \mathfrak{p}^n$ with $(b_1 b_2) \notin \mathfrak{p}$.
  Thus $(b_1 b_2)(a_1 \pm a_2) \in \mathfrak{p}^n$ yields $a_1 \pm a_2 \in \mathfrak{p}^{(n)}$.
- $a \in \mathfrak{p}^{(n)}$ & $c \in R \overset{?}{\Longrightarrow} ca \in \mathfrak{p}^{(n)}$.
  Pick $b \notin \mathfrak{p}$ with $ba \in \mathfrak{p}^n$. Then $b(ca) \in \mathfrak{p}^n$ & $b \notin \mathfrak{p} \Rightarrow ca \in \mathfrak{p}^{(n)}$.

Fix $a \in \mathfrak{p}^{(n+1)}$ & $b \notin \mathfrak{p}$ with $ba \in \mathfrak{p}^{n+1} \subseteq \mathfrak{p}^n$. Then, $a \in \mathfrak{p}^{(n)}$.

(2) • If $a \in \mathfrak{p}^n$, then $a = 1 \cdot a \in \mathfrak{p}^n$ & $1 \notin \mathfrak{p}$. Thus $a \in \mathfrak{p}^{(n)}$.
• If $a \in \mathfrak{p}^{(n)}$ then $ba \in \mathfrak{p}^n \subseteq \mathfrak{p}$ for some $b \notin \mathfrak{p}$. Thus $a \in \mathfrak{p}$.

(3) By (2) we have $\mathfrak{p} = \sqrt{\mathfrak{p}} = \sqrt{\mathfrak{p}^n} \subseteq \sqrt{\mathfrak{p}^{(n)}} \subseteq \sqrt{\mathfrak{p}} = \mathfrak{p}$, so $\sqrt{\mathfrak{p}^{(n)}} = \mathfrak{p}$.
To see that $\mathfrak{p}^{(n)}$ is $\mathfrak{p}$-primary we fix $ab \in \mathfrak{p}^{(n)}$ and assume $a \notin \mathfrak{p}^{(n)}$.
We want to show that $b \in \mathfrak{p}$. We argue by contradiction.
Since $ab \in \mathfrak{p}^{(n)}$ $\exists\, c \notin \mathfrak{p}$ with $cab = cba \in \mathfrak{p}^n$. Assuming that $b \notin \mathfrak{p}$, we get $cb \notin \mathfrak{p}$, so $a \in \mathfrak{p}^{(n)}$. This cannot happen by our assumption. Contradiction!

Thus, $b \in \mathfrak{P}$, as we wanted to show.

(4) We show the equality holds by proving the double inclusion.

($\subseteq$) Fix $\frac{b}{s} \in \mathfrak{P}^{(n)} R_{\mathfrak{P}}$ ie $s \notin \mathfrak{P}$ & $\exists\, c \notin \mathfrak{P}$ with $cb \in \mathfrak{P}^n$. Then

$$\frac{b}{s} = \frac{cb}{cs} \in \mathfrak{P}^n R_{\mathfrak{P}}.$$

($\supseteq$) Follows since $\mathfrak{P}^n \subseteq \mathfrak{P}^{(n)}$ so $\mathfrak{P}^n R_{\mathfrak{P}} \subseteq \mathfrak{P}^{(n)} R_{\mathfrak{P}}$. $\square$

<u>Proof of Krull's PIThm:</u> We fix $\mathfrak{P}$ a minimal prime over (a). If $\operatorname{codim} \mathfrak{P} = 0$, there is nothing to do.

Since localization is exact, we may assume R is local with maximal ideal $\mathfrak{P}$ (replace R with $R_{\mathfrak{P}}$, which is also Noetherian)

Assume $\operatorname{codim}_R \mathfrak{P} > 0$ & fix a chain $Q' \subseteq Q \subsetneq \mathfrak{P}$ prime ideals in R.

To show $\operatorname{codim} \mathfrak{P} \leq 1$ we must verify that $Q' = Q$. Taking the quotient of R by Q', we may assume $Q' = 0$ & R is a local Noetherian domain.

- We must show that $Q = 0$. For this, we consider the symbolic powers of Q. By Lemma 4 (1) $Q^{(n+1)} \subseteq Q^{(n)}$ $\forall n \in \mathbb{N}$

<u>Claim 1:</u> $Q^{(n)} \subset Q^{(n+1)} + (a)$ for some n

Pf/ $R/(a)$ is Noetherian & of dimension 0 because $\mathfrak{P}/(a)$ is both minimal (by hypothesis) & maximal ($(R,\mathfrak{P})$ is local $\Rightarrow (R/(a), \mathfrak{P}/(a))$ is local). Thus, $R/(a)$ is Artinian ie it satisfies the descending chain condition: $(Q^{(0)} + (a))/(a) \supseteq (Q^{(1)} + (a))/(a) \supseteq \cdots$.

Thus $\exists\, n$ st $Q^{(n)} + (a) = Q^{(n+1)} + (a)$ for some n & so $Q^{(n)} \subseteq Q^{(n+1)} + (a)$. $\square$

<u>Claim 2:</u> $Q^{(n)} = Q^{(n+1)} + \mathfrak{P} Q^{(n)}$ for some n

Pf/ ($\supseteq$) is clear

($\subseteq$) Pick n as in Claim 1. If $b \in Q^{(n)}$, write $b = c + ra$ for $c \in Q^{(n+1)}$ & $r \in R$. Then, $\boxed{ra = b - c \in Q^{(n)}}$ by Lemma 4 (1)

- Since $\mathfrak{p}$ is minimal over a, & $Q \subsetneq \mathfrak{p}$ we have $a \notin Q = \sqrt{Q^{(n)}}$ by Lemma 4
- Since $Q^{(n)}$ is Q-primary by Lemma 4 (3), we have $r \in Q^{(n)}$

Conclusion: $b = c + \underline{\underline{a}}r \in Q^{(n+1)} + \underline{\underline{\mathfrak{p}}}\, Q^{(n)}$, as we wanted. $\square$

- Take the quotient by $Q^{(n+1)}$ in Claim 2 to get:

$$Q^{(n)}/Q^{(n+1)} = \mathfrak{p}\, \frac{Q^{(n)}}{Q^{(n+1)}}$$

Since $Q^{(n)}$ is fg (R is Noetherian), so is $Q^{(n)}/Q^{(n+1)}$. Since $(R, \mathfrak{p})$ is local, Nakayama's Lemma implies $Q^{(n)}/Q^{(n+1)} = 0$, ie $\boxed{Q^{(n)} = Q^{(n+1)}}$.

- Localizing away from Q is exact, so $Q^{(n)} R_Q = Q^{(n+1)} R_Q$

By Lemma 4 (4) we have

$$Q^n R_Q = Q^{(n)} R_Q = Q^{(n+1)} R_Q = Q^{n+1} R_Q = (Q R_Q)(Q^n R_Q)$$

Again, Nakayama's Lemma applied to the fg R_Q-module $Q^n R_Q$ gives $Q^n R_Q = 0$.
Since R was integral, so is R_Q. Conclude: $Q^n = 0$, so $Q = 0$.