# Lecture XXXIV: Dimension Theory IV

**Last Time:** we discussed the following 2 theorems:

**Theorem 1** (Dimension of Polynomial Rings)

Fix $\mathbb{K}$ any field (not necessarily algebraically closed) & $n \in \mathbb{N}$. Then

(1) $\dim \mathbb{K}[x_1, \ldots, x_n] = n$

(2) All maximal chains of prime ideals in $\mathbb{K}[x_1, \ldots, x_n]$ have length $n$.
(w.r.t. inclusion)

**Theorem 2** (Krull's Principal Ideal Theorem) Let $R$ be Noetherian ring & let $a \in R$. Then, every minimal prime ideal $\mathfrak{P}$ over $(a)$ satisfies $\operatorname{codim}_R \mathfrak{P} \leq 1$.

## §1 More on Krull's Principal Ideal Theorem:

**Corollary 1:** Let $R$ be Noetherian ring & $a \in R$ be a non-zero divisor. Then for every minimal prime ideal $\mathfrak{P}$ over $(a)$ we have $\operatorname{codim}_R \mathfrak{P} = 1$.

**Proof:** We use the characterization of associated primes of $(0)$ from Lecture 6. Let $\mathfrak{P}_1, \ldots, \mathfrak{P}_r$ be the minimal primes of $R$ over $(0)$. By Proposition 2 §6.3:

$$\mathfrak{P}_i = \sqrt{(0 : b_i)} \qquad \text{for some } b_i \in R \smallsetminus \{0\}$$

($b_i \neq 0$ because $\mathfrak{P}_i$ is a proper ideal of $R$)

**Claim 1:** $a$ non-zero divisor $\Rightarrow a \notin \mathfrak{P}_i$ for all $i = 1, \ldots, r$

Pf/ Otherwise, $a \in \sqrt{(0 : b_i)}$ for some $i$. Thus $a^m b_i = 0$ for some $m \in \mathbb{N}$. Picking $m$ minimal we see $a^m b_i = a \underbrace{(a^{m-1} b_i)}_{\neq 0} = 0$ forcing $a$ to be a zero-divisor. Contradiction! $\square$

**Claim 2:** $\mathfrak{P}$ is not a minimal prime of $R$

Pf/ $a \in \mathfrak{P}$, $a \notin \mathfrak{P}_i$ $\forall i = 1, \ldots, r$. & $\operatorname{Min}(R) = \{\mathfrak{P}_1, \ldots, \mathfrak{P}_r\}$. $\square$

Thus $\mathfrak{P}_i \subsetneq \mathfrak{P}$ for some $i$. Therefore $\operatorname{codim}_R \mathfrak{P} \geq 1$.

By Krull's PID we have $\operatorname{codim}_R \mathfrak{P} \leq 1$, so equality holds.

**Example:** ① $R = \mathbb{R}[x,y]/(x^2+y^2-1)$ $\qquad$ $(x^2+y^2+1)$ is a non-zero divisor in $\mathbb{R}[x,y]$

$$\dim R \underset{\substack{\text{Thm 1 (2) \&} \\ (x^2+y^2-1) \text{ is prime}}}{=} \dim \mathbb{R}[x,y] - \operatorname{codim}_{\mathbb{R}[x,y]} (x^2+y^2+1) \underset{\substack{\text{Theorem 1} \\ \text{Corollary 1} \\ ((x^2+y^2+1) \text{ is minimal prime over } x^2+y^2+1)}}{=} 2-1 = 1.$$

② Same idea gives: $\dim \mathbb{R}[x,y]/(x^2+y^2+1) = 2-1 = 1$

But $V(x^2+y^2+1) = \emptyset$ in $\mathbb{A}^2_{\mathbb{R}}$. This is why we need to restrict to affine or quasi-affine varieties over $\overline{K} = K$

**Recall:** If $X \subseteq \mathbb{A}^n$ is a variety over $\overline{K} = K$, fix $I(X) \subseteq K[x_1, \dots x_n]$

We know it is a radical ideal. How to relate the components of $X$ with primes over $I(X)$?

**Proposition 1:** The irreducible components of $X$ correspond to the minimal primes over $I(X)$.

**Proof:** Since $K[x_1 \dots x_n]$ is Noetherian, we consider a minimal primary decomposition of $I(X)$. Say $I(X) = q_1 \cap \dots \cap q_s$

Since $I(X)$ is radical $I(X) = \sqrt{I(X)} = \sqrt{q_1} \cap \dots \cap \sqrt{q_s}$. & $\sqrt{q_1}, \dots \sqrt{q_s}$ are prime ideals

Thus, we only need to keep those $\sqrt{q_i}$ that are minimal primes over $\sqrt{I(X)}$.

$\Rightarrow$ $I(X) = \sqrt{I(X)} = \sqrt{q_{i_1}} \cap \dots \cap \sqrt{q_{i_r}}$ where $\sqrt{q_{i_j}}$ are the minimal primes over $I(X)$.

(The fact that we are not missing any minimal prime in the list was seen in Lemma 4 §5.2)

Then $X = V(I(X)) = V(\sqrt{q_{i_1}} \cap \dots \cap \sqrt{q_{i_r}}) = V(\sqrt{q_{i_1}} \cdots \sqrt{q_{i_r}}) = \bigcup_{j=1}^{r} V(\sqrt{q_{i_j}})$

Since $V(\sqrt{q_{i_j}})$ are irreducible, then we see that the irreducible components of $X$ correspond to minimal primes over $I(X)$ $\square$

**Theorem 3** (Krull) Let $X$ be a quasi-affine variety over $\overline{K} = K$, and fix $f \in \mathcal{O}(X)$. Then, any irreducible component $Y$ of $V_X(f) = \{u \in X \mid f(u) = 0\}$ has $\operatorname{codim}_X Y \leq 1$.

**Proof:** Fix assume $X = U \cap X'$ with $U \subseteq \mathbb{A}^n$ open & $X' \subseteq \mathbb{A}^n$ variety.

Set $R = \mathcal{O}(X) = \mathcal{O}_{X'}(X' \cap U)$

We prove the theorem by reducing to special cases:

**Claim 1:** We can assume $X$ is affine.

Pf/ Since $X$ is open in $X'$ & $\overline{Y} \subseteq X'$ is irreducible, Lemma 4 §31.2 yields

$$\operatorname{codim}_X Y = \operatorname{codim}_X \overline{Y} \cap X = \operatorname{codim}_{X'} \overline{Y}.$$

Furthermore $\overline{Y}$ is an irreducible component of $\overline{V_X(f)} = V_{X'}(f)$. $\square$

<u>Claim 2:</u> We can assume $X$ is irreducible

Pf/ If $X = X_1 \cup \cdots \cup X_s$, then $V_X(f) = \bigcup_{i=1}^{s} V_{X_i}(f)$

If $Y$ is a component of $V_X(f) \subseteq X$, then $Y \subseteq X_i$ for some $i = 1, \ldots, s$

( $Y = \bigcup_{i=1}^{s} (Y \cap X_i) \underset{Y \text{ irred}}{\Longrightarrow} Y \cap X_i = Y$ for some $i$ ) So $Y \subseteq V_{X_i}(f)$.

By Lemma 5 §31.2 $\operatorname{codim}_X Y = \max_{1 \leq i \leq s} \{ \operatorname{codim}_{X_i}(Y) : Y \subseteq X_i \}$

If $\operatorname{codim}_{X_i}(Y) \leq 1$ $\forall i$ with $Y \subseteq X_i$, the same will be true for $\operatorname{codim}_X Y$.

Now, we consider $R = \mathcal{O}(X) = \mathbb{K}[X] = \dfrac{\mathbb{K}[x_1, \ldots x_n]}{I(X)}$ if $X \subseteq \mathbb{A}^n$.

Thus, $R$ is a Noetherian ring

- By Proposition 1, any irreducible component $Y$ of $V_X(f)$ corresponds to a prime ideal $I_X(Y)$ of $R$ minimal over $f$.
- By Krull's Principal Ideal theorem : $\operatorname{codim}_Y X = \operatorname{codim}_R I_X(Y) \leq 1$

<u>Corollary 2:</u> Let $X$ be a quasi-affine variety over $\overline{\mathbb{K}} = \mathbb{K}$, and fix $f \in \mathcal{O}(X)$. Assume $f$ is a non-zero divisor. Then, every irreducible component $Y$ of $V_X(f)$ has $\operatorname{codim}_X Y = 1$.

Pf/ Follow the proof of Theorem 3 & use Corollary 1. instead of Krull's PI Theorem.

<u>Corollary 3:</u> Let $X$ be a quasi-affine variety over $\overline{\mathbb{K}} = \mathbb{K}$, and fix $f_1, \ldots, f_r \in \mathcal{O}(X)$. Then, any irreducible component $Y$ of $V_X(f_1, \ldots, f_r)$ has $\operatorname{codim}_X Y \leq r$.

<u>Proof:</u> Induction on $r$. • <u>Base case:</u> $r = 1$ is Theorem 3.

• <u>Inductive Step</u> We can follow the same reductions as in the proof of Theorem 3 & assume $X$ is irreducible & affine.

Set $X' = X \cap V(f_1, \ldots, f_{r-1})$ Then, $V_{X'}(f_r) = V_X(f_1, \ldots, f_r)$

Any irreducible component $Y$ of $V_{X'}(f_r)$ has $\operatorname{codim}_{X'} Y \leq 1$ by Theorem 3

But $X' = V_X(f_1, \ldots, f_{r-1})$ & $Y \subseteq V_{X'}(f_r)$ is irreducible forces $Y$ to lie in an irreducible component $Y'$ of $X'$. By Lemma 5 § 31.2 :

$$\operatorname{codim}_{X'} Y = \max_{1 \le i \le s} \{ \operatorname{codim}_{X_i'}(Y) : Y \subseteq X_i' \} \qquad (*)$$

where $X' = \bigcup_{i=1}^{s} X_i'$ is the irreducible decomposition of $X'$. Pick $Y' = X_i'$ realizing this maximum value.

We have $\underset{\text{irred}}{Y} \subseteq \underset{\text{irred}}{Y'} = X_i' \subseteq X' \subseteq \underset{\text{irred}}{X} \subseteq \mathbb{A}^n$. Furthermore

- $\operatorname{codim}_X Y' \le r-1$ by (IH)
- $\operatorname{codim}_{Y'} Y + \operatorname{codim}_X Y' = \operatorname{codim}_X Y$ because $X$ is assumed to be irreducible & every maximal chain of prime ideals in $\mathbb{K}[x_1, \ldots x_n]$ has length $n$ (We can extend the chain $I(Y) \supsetneq I(Y') \supsetneq I(X)$ to a maximal one)

Thus, $\operatorname{codim}_X Y = \operatorname{codim}_{Y'} Y + \operatorname{codim}_X Y' = \underbrace{\operatorname{codim}_{X'} Y}_{\le 1} + \underbrace{\operatorname{codim}_X Y'}_{\le r-1} \le r$, as we wanted to show. □

- We have a partial converse to this statement.

**Proposition 2:** Let $X$ be an affine variety over $\overline{\mathbb{K}} = \mathbb{K}$. If $Y$ is an irreducible closed subset of $X$ with $\operatorname{codim}_X Y = r \geq 1$, then there are $f_1, \ldots, f_r \in \mathcal{O}(X)$ such that $Y$ is an irreducible component of $V(f_1, \ldots, f_r)$.

⚠ We should not expect $Y = V(f_1, \ldots f_r)$.

We'll discuss the proof & ⚠ next time.