Lecture XXV: Dimension Theory V

Recall the following results from last time:

Theorem (Krull): Let X be a quasi-affine variety over $\overline{\mathbb{K}}=\mathbb{K}$, and fix $f \in \mathcal{O}(X)$. Then, any irreducible component Y of $V_X(f) = \{u \in X \mid f(u)=0\}$ has $\operatorname{codim}_X Y \leq 1$. Furthermore equality holds if f is a non-zero divisor on $\mathcal{O}(X)$.

Corollary: Let X be a quasi-affine variety over $\overline{\mathbb{K}}=\mathbb{K}$, and fix $f_1, \ldots, f_r \in \mathcal{O}(X)$. Then, any irreducible component Y of $V_X(f_1, \ldots, f_r)$ has $\operatorname{codim}_X Y \leq r$.

§1 More on Krull's Theorem:

- We have a partial converse to the corollary:

Proposition 1: Let X be an affine variety over $\overline{\mathbb{K}}=\mathbb{K}$. If Y is an irreducible closed subset of X with $\operatorname{codim}_X Y = r \geq 1$, then there are $f_1, \ldots, f_r \in \mathcal{O}(X)$ such that Y is an irreducible component of $V(f_1, \ldots, f_r)$.

Proof: Let $X_1, \ldots, X_s$ be the irreducible components of X.

- **Claim**: $\exists\, f_1 \in I_X(Y)$ with $X_i \not\subseteq V(f_1)$ $\forall i$

Pf/ We argue by contradiction. If this is not the case then for all $f \in I_X(Y)$ we have $X_i \subseteq V_X(f)$ for some i. Therefore, $f \in I(X_i)$ by the Nullstellensatz.

Thus
$$\underbrace{I_X(Y)}_{\text{prime}} \subseteq \bigcup_{i=1}^{N} \underbrace{I_X(X_i)}_{\text{prime}} \qquad (*)$$

Since $\operatorname{codim}_X Y = \max_{1\leq i\leq s} \{\operatorname{codim}_{X_i} Y : Y \subseteq X_i\} = r \geq 1$ by Lemma 5 §31.2 we conclude that $\operatorname{codim}_{X_i} Y \geq 1$ for all i with $Y \subseteq X_i$ (otherwise, $Y = X_i$ & $X_i \not\subseteq X_j$ if $j \neq i$ will give $r=0$) Thus $I_X(Y) \not\subseteq I_X(X_i)$ $\forall i=1,\ldots,s$

By Prime Avoidance, we cannot have $(*)$ Contradiction! □

- Now, choose f_1 as in Claim 1. Pick i with $Y \subseteq X_i$ with $\operatorname{codim}_{X_i} Y = r$

(such i exists by Lemma 5 § 31.2). Then $\underset{\text{irred}}{Y} \subseteq V_X(f_1) \subseteq X \Rightarrow Y \subseteq V_{X_i}(f_1) \subsetneq \underset{\text{irred}}{X_i}$

& f_1 is a non-zero divisor on $\mathcal{O}(X_i)$.

__Claim 2:__ $\operatorname{codim}_{V_{X_i}(f_1)} Y \leq r-1$

Pf/ Any irred component Y_{ij} of $V_{X_i}(f_1)$ has codim ≤ 1 in X_i

Since $\operatorname{codim}_{V_{X_i}(f_1)} Y = \max_{1\leq j\leq r_s} \{ \operatorname{codim}_{Y_{ij}} Y : Y \subseteq Y_{ij} \}$ $(**)$

$r \geq \operatorname{codim}_{X_i} Y = \operatorname{codim}_{Y_{ij}} Y + \underbrace{\operatorname{codim}_{X_i} Y_{ij}}_{=1 \text{ by Corollary 1}}$ whenever $Y \subseteq Y_{ij} \subseteq X_i$ are irreducible

Picking Y_{ij} realizing the max in $(**)$, we get

$$\operatorname{codim}_{V_{X_i}(f_1)} Y = \operatorname{codim}_{X_i} Y - 1 \underset{X_i \text{ is a comp of } X}{\leq} \operatorname{codim}_X Y - 1 = r-1 \qquad \square$$

__Note:__ $V_X(f_1) = \bigcup_{i=1}^{s} V_{X_i}(f_1)$.

We proceed by induction on r.

- __Base case:__ $r=1$

  Then $\operatorname{codim}_{V_{X_i}(f_1)} Y \leq 0$ yields $\operatorname{codim}_{V_{X_i}(f_1)} Y = 0$ ie Y is an irred component of $V_X(f_1)$

- __Inductive Step:__ set $s = \operatorname{codim}_{V_{X_i}(f_1)} Y$ & $X' = V_{X_i}(f_1) \subseteq X$

  . If $s=0$, then Y is an irred component of $V_{X_i}(f_1)$, hence one of $V_X(f_1)$

  Taking $f_2 = \cdots = f_r = f_1$ shows Y is an irred comp of $V_X(f_1, \ldots, f_r)$

  . If $s \geq 1$, by (IH) $\exists f_2, \ldots, f_s \in I_{X'}(Y) \subseteq I_X(Y)$ with Y an irred component of $V_{X'}(f_2, \ldots, f_s) = V_{X_i}(f_1, \ldots, f_s) = X_i \cap V(f_1 \ldots f_s)$

  . If $s = r-1$ this shows Y is an irred comp of $V_X(f_1 \ldots f_s)$

  . If $s < r-1$, take $f_s = f_{s+1} = \cdots = f_r$ to get $V_X(f_1 \ldots f_s) = V_X(f_1 \ldots f_r)$

and, thus Y is an irreducible component of $V_X(f_1, \ldots, f_r)$. $\square$

⚠ We cannot expect to find $f_1 \dots f_r$ with $Y = V_X(f_1, \dots, f_r)$, not even if we are willing to restrict to affine opens on both sides around any point of Y.

EXAMPLE $X = V(x_1x_2 - x_3x_4) \subseteq \mathbb{A}^4$ $\qquad Y = V(x_1, x_3)$

Both X & Y are irreducible

- $\dim Y = 2$ $\qquad \mathbb{A}^4 \supsetneq V(x_1) \supsetneq V(x_1, x_3)$ is maximal & $\operatorname{codim}_{\mathbb{A}^4} Y \leq 2$.
- $\dim X = 3$ $\qquad \operatorname{codim}_{\mathbb{A}^4} X = 1$ because $x_1x_2 - x_3x_4$ is a non-zero divisor in $\mathbb{K}[\mathbb{A}^4]$

$\Rightarrow \operatorname{codim}_X Y = 1$

Pick U open in X with $(0,0,0,0) \in U$.

Claim: $\nexists\, f \in \mathcal{O}(U)$ with $Y \cap U = V_U(f)$

Pf/ It's enough to prove this when U is a basic open $= D_X(g)$ $g \notin I(X)$.

so $Y \cap U = D_Y(g)$

If $Y \cap U = V_U(f)$ then $(x_1, x_3) = \sqrt{(f)}$ in $\mathbb{K}[X]_{(g)}$

More precisely $(x_1, x_3)_{(g)} = \sqrt{(f, x_1x_2 - x_3x_4)_{(g)}}$ in $\mathbb{K}[x_1, x_2, x_3, x_4]_{(g)}$.

CASE 1: $g \in \mathbb{K}$

- The condition $(f, x_1x_2 - x_3x_4) \subseteq (x_1, x_3)$ forces f to be of the form $f = Ax_1 + Bx_3$.
- For $\supseteq$ we have $\exists\, m \geq 1$ st $x_1^m \in (f, x_1x_2 - x_3x_4)$
$\qquad x_3^m \in (f, x_1x_2 - x_3x_4)$

$$x_1^m = h_1 f + h_1'(x_1x_2 - x_3x_4) = h_1(Ax_1 + Bx_3) + h_1'(x_1x_2 - x_3x_4)$$
$$x_3^m = h_3 f + h_3'(x_1x_2 - x_3x_4) = h_3(Ax_1 + Bx_3) + h_3'(x_1x_2 - x_3x_4)$$
in $\mathbb{K}[x_1, \dots, x_4]$

Setting $x_2 = x_4 = 0$ gives:

$$x_1^m = \tilde{h}_1(\tilde{A}x_1 + \tilde{B}x_3)$$
$$x_3^m = \tilde{h}_3(\tilde{A}x_1 + \tilde{B}x_3)$$
in $\mathbb{K}[x_1, x_3]$

Since $\mathbb{K}[x_1, x_3]$ is a UFD then $\tilde{A}x_1 + \tilde{B}x_3 = a x_1^r$
& $\tilde{A}x_1 + \tilde{B}x_3 = b x_3^s$ for $r, s \leq m$, $a, b \in \mathbb{K}$

The only option is $a=b=0$ $r=s=0$. But $\tilde{A}x_1+\tilde{B}x_3=0$ cannot happen since $x_1^m, x_3^m \neq 0$.

CASE 2: $g \in \mathbb{K}[x_1,\dots x_4] \setminus \mathbb{K}$ with $g(0,0,0,0)\neq 0$.

We argue similarly to case 1, incorporating powers of g into the (RHS) expressions.

- The condition $(f, x_1x_2-x_3x_4)_{(g)} \subseteq (x_1,x_3)_{(g)}$ forces f to be in $(x_1,x_3)_{(g)}$

Thus, $\exists\ k\geq 0$ with $g^k f = Ax_1+Bx_3$.

- For $\supseteq$ we have $\exists\ m\geq 1$ & $\ell \geq 0$ st. $g^\ell x_1^m \in (f, x_1x_2-x_3x_4)$
$g^\ell x_3^m \in (f, x_1x_2-x_3x_4)$.

In particular multiplying by g^k we get
$$g^{\ell+k}x_1^m \in (g^kf, g^k(x_1x_2-x_3x_4)) \subseteq (g^kf, x_1x_2-x_3x_4)$$
$$g^{\ell+k}x_3^m \in (g^kf, x_1x_2-x_3x_4)$$

Write $s=\ell+k\geq 0$. We can find $h_1, h_1', h_3, h_3' \in \mathbb{K}[x_1,x_2,x_3,x_4]$ such that:
$$g^s x_1^m = h_1(Ax_1+Bx_3)+h_1'(x_1x_2-x_3x_4)$$
$$g^s x_3^m = h_3(Ax_1+Bx_3)+h_3'(x_1x_2-x_3x_4)$$

Setting $x_2=x_4=0$ gives:
$$\tilde{g}^s x_1^m = \tilde{h}_1(\tilde{A}x_1+\tilde{B}x_3)$$
$$\tilde{g}^s x_3^m = \tilde{h}_3(\tilde{A}x_1+\tilde{B}x_3)$$
in $\mathbb{K}[x_1,x_3]$ and $\tilde{g}\in\mathbb{K}[x_1,x_3]\setminus\{0\}$.

Since $\tilde{g}(0,0)\neq 0$, we know that $\tilde{g}\notin \langle x_1,x_3\rangle$

- If $x_1 \nmid \tilde{A}x_1+\tilde{B}x_3$, then $x_1^m \mid \tilde{h}_1$ & so $\tilde{g}^s = \frac{h_1}{x_1^m}(Ax_1+Bx_3)\in\langle x_1,x_3\rangle$

Thus, $\tilde{g}\in\sqrt{\langle x_1,x_3\rangle}=\langle x_1,x_3\rangle$ Contra!

- Similarly $x_3\nmid \tilde{A}x_1+\tilde{B}x_3$ implies $\tilde{g}\in\langle x_1,x_3\rangle$ from the second equation. Contra!

In particular, $x_1x_3 \mid \tilde{A}x_1+\tilde{B}x_3$ in $\mathbb{K}[x_1,x_2,x_3,x_4]$ since $\mathbb{K}[x_1,\dots x_4]$ is a UFD

This in turn forces $x_3\mid\tilde{g}^s$ or $x_1\mid\tilde{g}^s$ ie $x_3\mid\tilde{g}$ or $x_1\mid\tilde{g}$. This cannot happen because $\tilde{g}\notin\langle x_1,x_3\rangle$. $\square$

§2 Dimension & regular dominant maps:

<u>Remark</u>: Fix $X \xrightarrow{\varphi} Y$ a regular map of quasi-affine varieties over $\overline{\mathbb{K}}=\mathbb{K}$ & $y \in Y$. Then, $f^{-1}(y)$ is a closed subset of Y. Thus, it is a quasi-affine variety. Same is true for $f^{-1}(Z)$ when Z is a closed subset of Y.

- Our goal is to determine $\dim(Z)$ for any closed subset Z of Y.

<u>Lemma 1</u>: If X is an irreducible quasi-affine variety & V is an open in X, we have
$$\dim X = \dim V$$

<u>Proof</u>: X & V is irreducible, so $\mathbb{K}(V) = \mathbb{K}(X)$ & $\dim V = \operatorname{tr\,deg}_{\mathbb{K}} \mathbb{K}(V)$, $\dim X = \operatorname{tr\,deg}_{\mathbb{K}} \mathbb{K}(X)$ by Corollary 2 § 33.2. □

- Fix a dominant map $X \xrightarrow{f} Y$ between irreducible quasi-affine varieties over $\overline{\mathbb{K}}=\mathbb{K}$ write $X = U \cap X'$, $Y = V \cap Y'$ with U, V open ($U \subseteq \mathbb{A}^n$, $V \subseteq \mathbb{A}^m$) and X', Y' irreducible affine varieties ($X' \subseteq \mathbb{A}^n$, $Y' \subseteq \mathbb{A}^m$).

Then $\mathbb{K}(X') = \mathbb{K}(X)$ & $\mathbb{K}(Y') = \mathbb{K}(Y)$ & f induces an injection
$\operatorname{Quot}(f^{\#}_{y}) : \mathbb{K}(Y) \hookrightarrow \mathbb{K}(X)$. Thus: $\boxed{\dim Y - \dim X = \operatorname{tr\,deg}_{\mathbb{K}(Y)} \mathbb{K}(X)}$

- We'll need the following result from § 31.2

<u>Lemma 4 §31.2</u>: Fix X topological space, $Y \subseteq X$ closed irreducible & $U \subseteq X$ open wit $U \cap Y \neq \phi$. Then $\operatorname{codim}_U (U \cap Y) = \operatorname{codim}_X (Y)$.

- Theorem 1 § 32.1 (finite surjective morphisms of affine varieties over $\overline{\mathbb{K}}=\mathbb{K}$ preserve dimensions) is consistent with the following results discussing how dimension interacts with fibers of regular dominant maps.

- <u>Theorem 1</u>: Let X, Y be irreducible quasi-affine varieties & let $f: X \longrightarrow Y$ be a dominant regular map. Let $Z \subseteq Y$ be closed & irreducible & W be an irreducible component of $f^{-1}(Z)$ that dominates Z. Then:

(1) $\operatorname{codim}_X W \leq \operatorname{codim}_Y Z$
(2) $\dim W \geq \dim Z + \operatorname{tr\,deg}_{\mathbb{K}(Y)} \mathbb{K}(X)$.

In particular, for every $y \in \operatorname{image}(f)$, all irreducible components of $f^{-1}(y)$ have dimension $\geq \dim X - \dim Y$.

Proof: Since X & W are irreducible in some $\mathbb{A}^n$ & $X = V \cap X'$ with V open in $\mathbb{A}^n$ & $W' \subseteq \mathbb{A}^n$ closed irred, we have

- $\operatorname{codim}_X W = \operatorname{codim}_{X'} \overline{W} = \dim X' - \dim \overline{W} = \dim X - \dim W$
- $\operatorname{codim}_Y Z = \dim Y - \dim Z$

Claim 1: (1) $\iff$ (2)

Pf/ Item (1) is equivalent $\dim X - \dim W \leq \dim Y - \dim Z$.

Rearranging things, we get $\dim W \geq \dim Z + \underbrace{\dim Y - \dim X}_{\operatorname{Tr\,deg}_{\mathbb{K}(Y)} \mathbb{K}(X)}$. $\square$

- It remains to show (1). Since W dominates Z, we have $\overline{f(W)} = Z$. Furthermore, given any U open in Y with $Z \cap U \neq \emptyset$, we have $f^{-1}(U) \cap W \neq \emptyset$. Thus,

$$\overline{f(\underbrace{f^{-1}(U) \cap W}_{\text{open dense in } W})}^{U} = \overline{f(\overline{f^{-1}(U) \cap W})} \cap U = \overline{f(W)} \cap U = Z \cap U$$

Claim 2: It suffices to prove the statement for $f^{-1}(U) \xrightarrow{f} U$.

Pf/ • The map $f^{-1}(U) \longrightarrow U$ is regular & dominant

- By our early discussion & Lemma 1, $\operatorname{Tr\,deg}_{\mathbb{K}(Y)} \mathbb{K}(X) = \operatorname{Tr\,deg}_{\mathbb{K}(U)} \mathbb{K}(f^{-1}(U))$
- By Lemma 4 §31.2, we have $\operatorname{codim}_{f^{-1}(U)} (f^{-1}(U) \cap W) = \operatorname{codim}_X (W)$

$\operatorname{codim}_Y Z = \operatorname{codim}_U (Z \cap U)$

Thus if (1) holds for $f^{-1}(U) \xrightarrow{f} U$, it will hold for $X \xrightarrow{f} Y$.

Claim 3: We may assume Y is affine.

Pf/ Write $Y = V \cap Y'$ where $Y' \subseteq \mathbb{A}^m$ affine irreducible & V is open in $\mathbb{A}^m$.

We pick $U = D_Y(g) = D(g) \cap Y$ in Claim 3 for some $g \in \mathbb{K}[\mathbb{A}^m]$.

Then $U \subseteq \mathbb{A}^{m+1}$ is affine via $U = V(I(Y')\mathbb{K}[x_1, \ldots x_{m+1}] + \langle 1 - g x_{m+1} \rangle)$ $\square$

Write $\boxed{s = \operatorname{codim}_Y (Z)}$

- If $s = 0$, then $Z = Y$ & so $W = X$ (all 4 sets are irred.) Thus, $\operatorname{codim}_X W = 0$.
- If $s \geq 1$, by Proposition 1 $\exists\, g_1, \ldots, g_s \in \mathcal{O}(Y)$ such that Z is an irreducible component of $V_Y(g_1, \ldots, g_s) = V_Y(\langle g_1, \ldots, g_s \rangle)$

Since $W \subseteq f^{-1}(Z)$ we have $W \subseteq f^{-1}(V_Y(\langle g_1, \ldots g_s \rangle)) = V(f^{\#}\langle g_1, \ldots, g_s \rangle)$

Set $\boxed{W' := V(f^{\#}(g_1), \ldots, f^{\#}(g_s))}$ $\quad f^{\#}(g_i) \in \mathcal{O}(X)$.

Claim 4: W is an irreducible component of W'.

Pf/ Pick W'' an irred comp of W' containing W. We must show $W = W''$.

Since $W \subseteq W'' \subseteq W'$ with W'' closed & irreducible, we have:

$$Z = \overline{f(W)} \subseteq \overline{f(W'')} \subseteq \overline{f(W')} = V(g_1, \ldots, g_s)$$

- But Z is an irred component of $V(g_1 \ldots, g_s)$ & $\overline{f(W'')}$ is irreducible. Therefore, $Z = \overline{f(W'')}$. In particular $W'' \subseteq f^{-1}(Z)$
- Since $W \subseteq \underset{\text{irred cc}}{W''} \subseteq f^{-1}(Z)$ & W is an irred component of $f^{-1}(Z)$, we get $W'' = W$.

By Corollary 3 §34.1, we get $\operatorname{codim}_X W \leq s$. This confirms (1).

- In the special case when $Z = \{y\}$ we get $\dim Z = 0$. By (2) $\dim W \geq \operatorname{tr\,deg}_{\mathbb{K}(Y)} \mathbb{K}(X)$. $\square$
- Our next statement establishes that $=$ holds on a dense open set of Y.

Theorem 2: With the conditions of Theorem 1, there exists an open subset V of Y such that $V \subseteq f(X)$ & for every irreducible, closed subset $Z \subseteq Y$ with $Z \cap V \neq \emptyset$, and every irreducible component W of $f^{-1}(Z)$ that dominates Z (meaning $\overline{f(W)} = Z$), we have (1) $\operatorname{codim}_X(W) = \operatorname{codim}_Y(Z)$

(2) $\dim W = \dim Z + \operatorname{tr\,deg}_{\mathbb{K}(Y)} \mathbb{K}(X)$

In particular, for every $y \in V$, every irreducible component of $f^{-1}(y)$ has dimension $\dim X - \dim Y$.

Proof: As in the proof of Theorem 1, we have (1) $\Leftrightarrow$ (2), so we only need to prove (2)

By Theorem 1 we need only show $\dim W \leq \dim Z + \operatorname{tr\,deg}_{\mathbb{K}(Y)} \mathbb{K}(X)$.

Using Claims 2 & 3 from the same proof, we can assume Y is affine.

Claim 1: We can assume X is affine

Pf/ Write an open cover of X by affine opens in X: $X = U_1 \cup \cdots \cup U_m$ (pick $U_i = D_X(g_i)$)

Since X & Y are irreducible, we can restrict to $f_i : U_i \longrightarrow Y$, which is also regular & dominant. Since X is irreducible, then U_i is irreducible as well.

If we find $V_i \subseteq Y$ non-empty open for each f_i, then $V = \bigcap_{i=1}^{m} V_i$ is open in Y

- $V \neq \emptyset$ because Y is irreducible

- $W = \bigcup_{i=1}^{m} (W \cap U_i)$ satisfies • $W_i = W \cap U_i$ is an irreducible component of $f_i^{-1}(Z)$ & it dominates Z.

By Lemma 4 §31.2, $\operatorname{codim}_{U_i} W_i = \operatorname{codim}_X W$ as we want to show.

$\underset{\text{choice of } V_i \to}{\parallel}$ $\operatorname{codim}_Y Z$

□

- Since $X \xrightarrow{f} Y$ is reg dominant with X, Y affine irreducible, we consider $f^{\#}_Y : \mathbb{K}[Y] \hookrightarrow \mathbb{K}[X]$ the induce homomorphism of rings (injective since f is dominant)

- We consider the $\mathbb{K}(Y)$-algebra : $\boxed{S := \mathbb{K}[X] \otimes_{\mathbb{K}[Y]} \mathbb{K}(Y)}$

- Properties of S: • S is a domain ($\mathbb{K}[X] \subseteq S \subseteq \mathbb{K}(X)$) & $\mathbb{K}(Y) \subseteq S$.
  - $\operatorname{Quot}(S) = \mathbb{K}[X]$
  - $\operatorname{Tr\,deg}_{\mathbb{K}(Y)} \operatorname{Quot}(S) = \operatorname{Tr\,deg}_{\mathbb{K}(Y)} \mathbb{K}(X) =: r$

By Noether Normalization, we can find $y_1, \dots, y_r \in S$ algebraically independent over $\mathbb{K}(Y)$ such that $\alpha : \mathbb{K}(Y)[y_1, \dots y_r] \hookrightarrow S$ is finite.

Replacing each y_i by $a_i y_i$ for some $a_i \in \mathbb{K}[Y]$, we may assume $y_i \in \mathbb{K}[X]$ for all $i = 1, \dots, r$.

<u>Claim 2:</u> $\exists$ $s \in \mathbb{K}[Y] \setminus \{0\}$ such that

$$\varphi : \mathbb{K}[Y]_{(s)}[y_1, \dots y_r] \hookrightarrow \mathbb{K}[X]_{(f^{\#}(s))}$$

is finite

Pf/ Write $\mathbb{K}[X] = \dfrac{\mathbb{K}[x_1, \dots x_n]}{I(X)}$ & view $\bar{x}_1, \dots, \bar{x}_n$ as generators of $\mathbb{K}[X]$ as a $\mathbb{K}$-algebra. By construction $\bar{x}_i \in S$ $\forall i = 1, \dots, n$. We assume $\bar{x}_i \neq 0$, $\forall i$

Since α is finite, we know $\bar{x}_i$ satisfies a monic equation:

$$\bar{x}_i^{m_i} + a_{i,1} \bar{x}_i^{m_i - 1} + \dots + a_{i, m_i} = 0 \qquad \text{for some } m_i \geq 0$$

with $a_{i,j} \in \mathbb{K}(Y)[y_1 \cdots y_r]$

Pick $s \in \mathbb{K}[Y]\setminus\{0\}$ with $s\,a_{i,j} \in \mathbb{K}[Y][y_1,\dots,y_r]$ $\forall i,j$.
Then $\overline{x}_i$ is integral over $\mathbb{K}[Y]_{(s)}[y_1,\dots,y_r]$ $\forall i=1,\dots,n$

<u>Conclusion</u>: $\varphi : \mathbb{K}[Y]_{(s)}[y_1,\dots,y_r] \hookrightarrow \mathbb{K}[X]_{(f^\#(s))}$ is finite

<u>Recall</u> $\mathcal{O}(D_Y(s)) = \mathbb{K}[Y]_{(s)}$ & $\mathcal{O}(D_X(f^\#(s))) = \mathbb{K}[X]_{(f^\#(s))}$
by Theorem 1 §28.1 ($\overline{\mathbb{K}} = \mathbb{K}$)

Replacing $f : X \to Y$ by $D_X(f^\#(s)) \xrightarrow{f} D_Y(s)$ which is also regular & dominant between affine varieties, we use the map φ from Claim 2 to factor f via

$$\begin{tikzcd} & Y \times \mathbb{A}^r \arrow[dr, two heads, "p_1"] & \\ X \arrow[ur, "g"] \arrow[rr, "f"'] & & Y \end{tikzcd}$$

$g^\# = \varphi$ is finite & g is dominant so g is surjective by Theorem 1 §29.1
Therefore, $f = p_1 \circ g$ is surjective

- Picking Z & W as in the statement, we get $g(W) \subseteq Z \times \mathbb{A}^r$.

- Theorem 1 § 32.1 gives $\dim W = \dim g(W)$
- Lemma 1 § 31.2 gives $\dim g(W) \leq \dim(Z \times \mathbb{A}^r) \underset{\text{HW7}}{=} \dim Z + \underbrace{\dim \mathbb{A}^r}_{=r \text{ by Theorem 1 (2) §33.2}}$

<u>Conclusion</u>: $\dim W \leq \dim Z + r$, as we wanted.

We end by discussing an important corollary:

<u>Corollary 1</u>: If $f : X \to Y$ is a morphism of quasi-affine varieties such that all fibers of f have dimension r (in particular, f is surjective), then $\dim X = \dim Y + r$.
<u>Proof</u>: If $Y_1,\dots,Y_m$ are the irreducible components of Y, each morphism $f^{-1}(Y_i) \to Y_i$ has all fibers of dimension r. Since $\dim X = \max\limits_{1\leq i\leq m} \dim f^{-1}(Y_i)$ and $\dim Y = \max\limits_{1\leq i\leq m} \dim Y_i$ by Lemma 2 §31.2, then it suffices to prove the statement when Y is irreducible.

Let $X = X_1 \cup \dots \cup X_s$ be the irreducible decomposition of X, & write
$$d_i = \dim X_i - \dim\left(\overline{f(X_i)}\right)$$

• By Theorem 2 there is an open subset V_i of $\overline{f(X_i)}$ with $V_i \subseteq f(X_i)$ such that every fiber of $f_i : X_i \longrightarrow \overline{f(X_i)}$ over a point in V_i has dimension d_i.

We prove the statement by checking that both inequalities hold.

<u>Remark</u>: Note that the set $\bigcap_{i=1}^{m} V_i \neq \emptyset$ because Y is irreducible (we can prove this by induction on m combined with the fact that irreducible sets are connected)

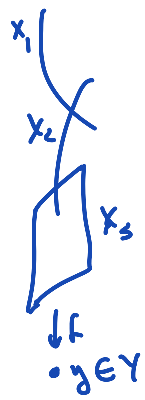
($\leq$) By construction $d_i \leq r$ for all i. In particular:

$$\dim(X_i) = d_i + \dim\left(\overline{f(X_i)}\right) \leq r + \dim\left(\overline{f(X_i)}\right) \leq r + \dim Y$$

by Lemma 1 § 31.2.

($\geq$) To show $\geq$ we must find a suitable i_0 with $\dim(X_{i_0}) \geq r + \dim Y$.

Pictorially:



We want to pick i_0 such that $\dim f_{i_0}^{-1}(y)$ is maximal

By construction: $\dim f^{-1}(y) = \max_{1 \leq i \leq m} f_i^{-1}(y)$ when $y \in \bigcap_{i=1}^{m} V_i$. Thus $\exists\, i_0$ with $d_{i_0} = r$.

Furthermore, we have:

<u>Claim</u>: $\overline{f(X_{i_0})} = Y$

Pf/ Since $V_{i_0} \subseteq f(X_{i_0}) \subseteq Y$ & V_{i_0} is open in Y which is irreducible, it follows that $Y = \overline{V_{i_0}} \subseteq \overline{f(X_{i_0})} \subseteq Y$, ie $\overline{f(X_{i_0})} = Y$. $\square$

Thus $\dim(X_{i_0}) = \dim Y + d_{i_0} = \dim Y + r$. Since $\dim(X_{i_0}) \leq \dim X$ by Lemma 1 § 31.2, we have $\dim X \geq \dim Y + r$, as we wanted to show. $\square$