Lecture XXXVI: Tangent spaces

Next goals: ① Define the notion of smoothness

② Study singularities of abstract varieties over $\overline{\mathbb{K}} = \mathbb{K}$.

§1. Vector spaces from Noetherian local rings:

Fix a Noetherian local ring $(R, \mathfrak{m})$ & let $\mathbb{K} = R/\mathfrak{m}$ be the residue field

Lemma 1: The set $\mathfrak{m}/\mathfrak{m}^2$ is a $\mathbb{K}$-vector space of finite dimension. Moreover
$\dim_{\mathbb{K}} \mathfrak{m}/\mathfrak{m}^2$ = minimal number of generators of the ideal $\mathfrak{m}$

Proof: We define the action of $\mathbb{K}$ on the ab. group $\mathfrak{m}/\mathfrak{m}^2$. Given $a \in R$, $m \in \mathfrak{m}$ we set:

$$\bar{a}\cdot\bar{m} = (a+\mathfrak{m})\,(m+\mathfrak{m}^2) = am + \underbrace{m\,\mathfrak{m} + a\,\mathfrak{m}^2 + \mathfrak{m}^3}_{\subseteq \mathfrak{m}^2} = am + \mathfrak{m}^2 = \overline{am}.$$

This action satisfies the axioms for a vector space over $\mathbb{K}$.

- Since R is Noetherian, $\mathfrak{m}$ is finitely generated say by $\{m_1, \ldots m_n\}$
- Then $\mathfrak{m}/\mathfrak{m}^2$ is generated over $\mathbb{K}$ by $\langle \bar{m}_1, \ldots, \bar{m}_n \rangle$ since $m = \sum_{i=1}^n r_i m_i$ yields $\bar{m} = \sum_{i=1}^n \overline{r_i m_i} = \sum_{i=1}^n \bar{r}_i \cdot \bar{m}_i$

  Thus $\dim_{\mathbb{K}} \mathfrak{m}/\mathfrak{m}^2 \leq n$ (*)
- To characterize $\dim_{\mathbb{K}} \frac{\mathfrak{m}}{\mathfrak{m}^2} = r$ we pick a basis $\{\bar{m}'_1, \ldots, \bar{m}'_r\}$ with $m'_i \in \mathfrak{m}$

  Then, $I = \langle m'_1, \ldots, m'_r \rangle \subseteq \mathfrak{m}$ satisfies $\boxed{\mathfrak{m} = I + \mathfrak{m}^2}$

Thus $M = \mathfrak{m}/I$ is a f.g module over R & $\mathfrak{m}\left(\mathfrak{m}/I\right) = \mathfrak{m}/I$ by $\boxed{}$. By Nakayama's lemma $M = 0$ ie $\mathfrak{m} = I$. Thus, $\mathfrak{m}$ can be generated by r elements.
The number is minimal by (*).

§1. Tangent spaces:

Fix $(X, \mathcal{O}_X)$ an algebraic variety over $\mathbb{K} = \overline{\mathbb{K}}$ & let $p \in X$ be a point.
By definition $\exists\, U \subseteq X$ open affine with $p \in U$. Then $\mathcal{O}_{X,p} = \mathcal{O}_{U,p}$.
Write $U \simeq Y \subseteq \mathbb{A}^n$ where Y is an affine variety. Then, by Theorem 1 §28.1, we have that
$\mathcal{O}_{X,p} \simeq \mathcal{O}_{Y,p} \simeq \mathbb{K}[Y]_{m_p}$ is a Noetherian local ring with maximal ideal m_p.

Furthermore by the Nullstellensatz $m_p = \langle y_1 - p_1, \dots, y_n - p_n \rangle$ & $\dfrac{\mathbb{K}[y]_{m_p}}{m_p \mathbb{K}[y]_{m_p}} \cong \mathbb{K}$

Definition: The *tangent space* of X at p is the $\mathbb{K}$-vector space

$$T_p X := \left(m_p / m_p^2 \right)^\vee = \mathrm{Hom}_{\mathbb{K}} \left(m_p / m_p^2 \,;\, \mathbb{K} \right)$$

Remark: The construction of $T_p X$ is local so we may always assume $(X, \mathcal{O}_X)$ is affine

Proposition 1: If $X \subseteq \mathbb{A}^n$ is an affine variety & $p \in X$, then

$$T_p X \cong \left\{ \underline{x} \in \mathbb{K}^n : \sum_{i=1}^n \frac{\partial f}{\partial x_i}(p)\, x_i = 0 \quad \forall f \in I(X) \right\} \quad \text{as } \mathbb{K}\text{-vector spaces}$$

Moreover if $I(X) = \langle f_1, \dots, f_s \rangle$, then $(\mathrm{RHS}) = \bigcap_{j=1}^{s} \left\{ \underline{x} \in \mathbb{K}^n : \sum_{i=1}^n \frac{\partial f_j}{\partial x_i}(p) x_i = 0 \right\}$

Name: We call the (RHS) the *embedded tangent space* of X in $\mathbb{A}^n$.

Proof: Write $I(X) = \langle f_1, \dots, f_s \rangle$ & $p = (p_1, \dots, p_n) \in X$.

$$\mathcal{O}_{X,p} = \left(\frac{\mathbb{K}[x_1, \dots x_n]}{\langle f_1, \dots f_s \rangle} \right)_{m_p} \cong \frac{\mathcal{O}_{\mathbb{A}^n, p}}{\langle f_1, \dots f_s \rangle \mathcal{O}_{\mathbb{A}^n, p}}$$

$$m_p = \langle x_1 - p_1, \dots, x_n - p_n \rangle\, \mathcal{O}_{\mathbb{A}^n, p} / \langle f_1, \dots f_s \rangle$$

Therefore $\boxed{\mathcal{O}_{X,p} / m_p^2 \cong \dfrac{\mathbb{K}[x_1, \dots x_n]}{\langle f_1, \dots f_s \rangle + \langle x_1 - p_1, \dots, x_n - p_n \rangle^2}}$ $(*)$

- If $f \in \mathbb{K}[x_1, \dots x_n]$, we can write f as

$$f = f(p) + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(p)(x_i - p_i) \mod \langle x_1 - p_1, \dots, x_n - p_n \rangle^2$$

If $f \in I(X)$, we get $f(p) = 0$ so

$$\langle f_1 \dots f_s \rangle + \langle x_1 - p_1, \dots, x_n - p_n \rangle^2 = \left\langle \sum_{i=1}^n \frac{\partial f}{\partial x_i}(p)(x_i - p_i) \ \forall f \in I(X) \right\rangle + \langle x_1 - p_1, \dots, x_n - p_n \rangle^2$$

Setting $e_i := \overline{x_i - p_i} \in m_p / m_p^2$, we see that $\{e_1, \dots e_n\}$ is a basis for m_p / m_p^2 (if n is minimal)

Thus $(*)$ yields: $\boxed{\dfrac{m_p}{m_p^2} = \langle e_1, \dots, e_n \rangle \Big/ \left\langle \sum_{i=1}^n \frac{\partial f}{\partial x_i}(p)\, e_i \ , \forall f \in I(X) \right\rangle}$

$\Rightarrow T_p X = \mathrm{Hom}_{\mathbb{K}}\left(\frac{m_p}{m_p^2}, \mathbb{K}\right) \cong \{ \underline{x} \in \mathbb{K}^n : \sum_{i=1}^n \frac{\partial f}{\partial x_i}(p)\, x_i = 0 \ \forall f \in I(X) \}$

$\varphi \longmapsto (x_i := \varphi(e_i))_{i=1}^n$

- In particular, if $f = \sum_{j=1}^n g_j f_j$, then $\frac{\partial f}{\partial x_i} = \sum_{j=1}^n \frac{\partial g_j}{\partial x_i} f_j + \sum_{j=1}^n g_j \frac{\partial f_j}{\partial x_i}$

Evaluating at $\underline{x} = p$ gives $\frac{\partial f}{\partial x_i}(p) = \sum_{j=1}^n g_j \frac{\partial f_j}{\partial x_i}(p)$

Thus, if $f \in I(X)$ we have. $\sum_{i=1}^n \frac{\partial f}{\partial x_i}(p)\, x_i = \sum_{j=1}^n g_j \sum_{i=1}^n \frac{\partial f_j}{\partial x_i}(p)\, x_i$

This proves the last claim of the statement. □

Examples: ① $X = \mathbb{A}^n$ $\quad m_p = \langle x_1 - p_1, \ldots, x_n - p_n \rangle \quad m_p / m_p^2 \simeq \langle e_1, \ldots, e_n \rangle \simeq \mathbb{K}^n$

so $T_p X \simeq (\mathbb{K}^n)^\vee = \mathbb{K}^n$ & (RHS) of Prop is also $\mathbb{K}^n$ $\quad (I(X) = \{0\})$

m_p is minimally generated by n elements, so $\dim_{\mathbb{K}} T_p X = \dim_{\mathbb{K}} m_p / m_p^2 = n$.

② $X = \{p\}$ $\quad m_p = \langle 0 \rangle \mathcal{O}_{X,p}$ $\quad$ Thus $m_p / m_p^2 = \{0\}$. $\quad T_p X \simeq \{0\}$.

Using Proposition 1, we can verify this. We generate $I(X)$ with $\langle x_1 - p_1, \ldots, x_n - p_n \rangle$.

$f_j = x_j - p_j \rightsquigarrow \frac{\partial (x_j - p_j)}{\partial x_i}(p) = \delta_{ij}$ $\quad$ Thus, $\sum_{i=1}^n \frac{\partial f_j}{\partial x_i}(p)\, x_i = x_j$

$T_p X = \{ \underline{x} : x_1 = \cdots = x_n = 0 \} = \{0\}$.

③ $X = V(f)$ $\quad m_p = \langle x_1 - p_1, \ldots, x_n - p_n \rangle \supseteq \langle f \rangle$

$T_p X = \{ \underline{x} : \sum_{i=1}^n \frac{\partial f}{\partial x_i}(p)\, x_i = 0 \} = (\nabla f_p)^\perp$.

Example: $f = x^3 - y^2$ $\quad T_0 X = \mathbb{K}^2$ has dim 2; $T_p X = (\nabla f_p)^\perp$ for $p \neq \underline{0}$ has dim 1

$\dim V(f) = 1$.

These inequalities are no accident. During the next lectures, we will prove:

Theorem 1: $\dim_{\mathbb{K}} T_p X \geq \dim_p X$.

Thus, a jump in dimension will detect singularities

Definition: A point $p \in X$ is nonsingular (or smooth) if $\dim_{\mathbb{K}} T_p X = \dim_p X$

Otherwise, it is singular.

The variety X is nonsingular (or smooth) if all its points are smooth.

Q: How do tangent spaces relate under regular maps?

Proposition 2: If $f: X \longrightarrow Y$ is a regular morphism of varieties and $p \in X$, then f induces a linear map $df_p : T_pX \longrightarrow T_{f(p)}Y$. This assignment is functorial.

Proof: The morphism of sheaves: $f^\# : \mathcal{O}_Y \longrightarrow f_*(\mathcal{O}_X)$ induces a morphism of local rings

$$\varphi = f^\#_{f(p)} : \underset{\substack{\cup\!\mathsf{I} \\ m_{f(p)}}}{\mathcal{O}_{Y,f(p)}} \longrightarrow \underset{\substack{\cup\!\mathsf{I} \\ m_p = f_y^\#(m_{f(p)})}}{\mathcal{O}_{X,p}}$$

In turn we get a $\mathbb{K}$-linear homomorphism $\overline{\varphi} : m_{f(p)}/m^2_{f(p)} \longrightarrow m_p/m_p^2$

Taking duals gives a linear map
$$df_p : T_pX \longrightarrow T_pY$$
$$\psi \longmapsto \psi \circ \overline{\varphi}$$

Claim: The assignment $f \longmapsto df_p$ is functorial

Pf/ (1) $d\,id_p = id_{T_pX}$ since $\varphi = id \Rightarrow \overline{\varphi} = id$.

(2) Consider $X \xrightarrow{f} Y \xrightarrow{g} Z$ Then: $d(g\circ f)_p = dg_{f(p)} \circ df_p$

This is so because the map on stalks is functorial & dualizing is also functorial.

Lemma 2: If $Y \subseteq X$ is a closed variety & $i: Y \hookrightarrow X$ is the inclusion, then for every $p \in Y$, the linear map $di_p : T_pY \longrightarrow T_pX$ is injective

Proof: $Y \hookrightarrow X$ implies $i^\# : \mathcal{O}_X \longrightarrow \mathcal{O}_Y$ is surjective. Thus $\varphi = i^\#_p : \mathcal{O}_{X,p} \longrightarrow \mathcal{O}_{Y,p}$ is surjective, and thus so is $\overline{\varphi}$.

Dualizing the surjective map gives an injection
$$di_p : T_pY \longrightarrow T_pX$$
$$\Phi \longmapsto \Phi \circ \overline{\varphi}$$

Corollary 1: If $Y \subseteq \mathbb{A}^n$ is a closed subvariety, then the image of the map $di_p : T_pY \longrightarrow T_p\mathbb{A}^n \simeq \mathbb{K}^n$ is the embedded tangent space of Y at p.

Q: Can we write df_p in terms of bases of T_pX & $T_{f(p)}Y$ given in Proposition 1?

Proposition 3. Assume $X \subseteq \mathbb{A}^m$, $Y \subseteq \mathbb{A}^n$ & $f = (f_1, \ldots, f_n) : X \longrightarrow Y$ is a polynomial map. Then, $df_p : T_pX \longrightarrow T_{f(p)}Y$ corresponds to the multiplication map $\mathbb{K}^m \xrightarrow{J} \mathbb{K}^n$ by the Jacobian matrix $\left(\frac{\partial f_i}{\partial x_j}(p)\right)_{i,j}$ restricted to the embedded tangent spaces in $\mathbb{K}^m$ & $\mathbb{K}^n$ (see Proposition 1)

Proof: (1) We first prove the statement for $X=\mathbb{A}^m$ $\quad Y=\mathbb{A}^n$.

Then $\varphi = f^{\#}_{f(p)} : \mathbb{K}[y_1 \dots y_n]_{m_{f(p)}} \longrightarrow \mathbb{K}[x_1 \dots x_m]_{m_p}$ $\qquad m_{f(p)} = \langle y_1 - f_1(p), \dots, y_n - f_n(p)\rangle$

$$y_i \longmapsto f_i(\underline{x})$$
$$\mathbb{K} \longmapsto \mathbb{K}$$

Thus $\quad \varphi(y_i - f_i(p)) = f_i(\underline{x}) - f_i(p)$.

Writing the Taylor expansion of f_i around p gives

$$f_i(\underline{x}) - f_i(p) = \sum_{j=1}^{m} \frac{\partial f_i}{\partial x_j}(p)\,(x_j - p_j) \mod m_p^2$$

Thus $\quad \overline{\varphi}(y_i - f_i(p)) = \sum_{j=1}^{m} \frac{\partial f_i}{\partial x_j}(p)\,(\overline{x_j - p_j})$ & so

$$df_p : T_p\mathbb{A}^m \longrightarrow T_p\mathbb{A}^n$$
$$(z_1, \dots z_m) \longmapsto \left[\frac{\partial f_i}{\partial x_j}(p)\right]_{i,j} \begin{bmatrix} z_1 \\ \vdots \\ z_m \end{bmatrix} \mod m_{f(p)}^2$$

(2) Since f is a polynomial map, it induces a morphism $\mathbb{A}^n \xrightarrow{\tilde{f}} \mathbb{A}^m$

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[drr, "f"'] & \mathbb{A}^n \arrow[r, "\tilde{f}"] & \mathbb{A}^m \\ & & Y \arrow[u, hook, "j"'] \end{tikzcd}$$

with image $(\tilde{f} \circ i) \subseteq Y$.

By functoriality,

$$\begin{tikzcd} df : T_pX \arrow[r, "df_p"] \arrow[d, hook, "i_p"'] \arrow[dr, phantom, "\circlearrowleft"] & T_{f(p)}Y \arrow[d, "dj_{f(p)}"] \\ T_p\mathbb{A}^n \arrow[r, "\tilde{df_p}"] & T_{f(p)}\mathbb{A}^m \end{tikzcd}$$

The vertical maps correspond to the identifications of Proposition 1. The bottom map has representing matrix given by the Jacobian matrix at p. $\square$

Q: What happens if f is rational?

A: We'll see this next time, after we discuss tangent spaces to projective varieties.