Lecture XXXVII: Projective Tangent spaces & smooth varieties

Recall: Given X an algebraic variety over $\mathbb{K}=\overline{\mathbb{K}}$ & $p\in X$, pick U affine open with $p\in X$

$T_pX = T_pU := \left(\mathfrak{m}_p/\mathfrak{m}_p^2\right)^\vee = \mathrm{Hom}_{\mathbb{K}}\left(\mathfrak{m}_p/\mathfrak{m}_p^2, \mathbb{K}\right)$ $\qquad \mathfrak{m}_p \subseteq \mathcal{O}_{X,p} = \mathcal{O}_{U,p}$! maxl ideal

($\dim_{\mathbb{K}} \mathfrak{m}_p/\mathfrak{m}_p^2$ = minimal number of generators of $\mathfrak{m}_p$ in $\mathcal{O}_{X,p}$ ($\leq n$ if $U\subseteq \mathbb{A}^n$)

Name: T_pX = Tangent space to X at p

Proposition: ① If $X\subseteq\mathbb{A}^n$ is an affine variety & $p\in X$, then

$$T_pX \simeq \left\{ \underline{x}\in\mathbb{K}^n : \sum_{i=1}^n \frac{\partial f}{\partial x_i}(p)\, x_i = 0 \quad \forall f\in I(X)\right\} \quad \text{as } \mathbb{K}\text{-vector spaces}$$

Moreover if $I(X) = \langle f_1,\ldots,f_s\rangle$, then $(\mathrm{RHS}) = \bigcap_{j=1}^s \left\{\underline{x}\in\mathbb{K}^n : \sum_{i=1}^n \frac{\partial f_j}{\partial x_i}(p)\, x_i = 0\right\}$

② If is a $X \xrightarrow{f} Y$ (polynomial) morphism of affine varieties then we have a natural linear map $df_p : T_pX \longrightarrow T_{f(p)}Y$. This assignment is functorial. If $X\subseteq\mathbb{A}^n$, $Y\subseteq\mathbb{A}^m$, it comes from restricting $J:\mathbb{K}^n\longrightarrow\mathbb{K}^m$ where $J(\underline{x}) = (\mathrm{Jac}(\underline{f},p))$ to $T_pX\subseteq\mathbb{K}^n$.

Q: What if f is regular or rational?

A: Same thing! To see this, we go to projective space!

§1 Tangent spaces for projective varieties:

Fix $X\subseteq\mathbb{P}^n$ projective variety over $\overline{\mathbb{K}}=\mathbb{K}$ & $p\in X$. **Recall:** $I(X)\subseteq\mathbb{K}[x_0,\ldots,x_n]$ is homogeneous

Definition: The projective tangent space of X at p is the linear subspace of $\mathbb{P}^n$

$$T_pX := \left\{\underline{x}\in\mathbb{P}^n : \sum_{j=0}^n \frac{\partial f}{\partial x_j}(p)\, x_j = 0 \quad \forall f\in I(X) \text{ homogeneous}\right\} \subseteq \mathbb{P}^n$$

Lemma 1: (1) T_pX is a linear subspace of $\mathbb{P}^n$ & $p\in T_pX$

(2) $T_pX = \bigcap_{i=1}^r \left\{\underline{x}\in\mathbb{P}^n \quad \sum_{j=0}^n \frac{\partial f_i}{\partial x_j}(p)\, x_j = 0\right\}$ where $f_1,\ldots,f_r$ are homogeneous polynomials generating $I(X)$.

Proof: The (RHS) is well-defined. Indeed, since f is homogeneous, so is $\frac{\partial f}{\partial x_j}$ (its degree is $\deg f - 1$). So $\frac{\partial f}{\partial x_j}(\lambda p) = \lambda^{d-1}\frac{\partial f}{\partial x_j}(p)$

Hence $\sum_{j=0}^{n} \frac{\partial f}{\partial x_j}(\lambda p)\, x_j = \lambda^{d-1} \sum_{j=0}^{n} \frac{\partial f}{\partial x_j}(p)\, x_j$, so the (RHS) is independent of the choice of representative for $p \in X$.

- Euler's identity $\left(\deg f \cdot f = \sum_{j=0}^{n} \frac{\partial f}{\partial x_j} x_j \text{ for any } f \text{ homogeneous}\right)$ confirms that $p \in T_pX$
- For the second part, note that if $f = \sum_{j=1}^{r} g_j f_j$ with $\langle f_1, \dots f_r\rangle$ as in the statement & $f \in I(X)$ is homogeneous, then we can assume g_j's are also homogeneous. The product rule combined with the condition $f_j(p) = 0 \;\forall p$ gives
$$\sum_{i=0}^{n} \frac{\partial f}{\partial x_i}(p)\, x_i = \sum_{i=0}^{n} \sum_{j=1}^{r} g_j(p) \frac{\partial f_j}{\partial x_i}(p)\, x_i = \sum_{j=1}^{r} g_j(p) \left(\sum_{i=0}^{n} \frac{\partial f_j}{\partial x_i}(p)\, x_i\right)$$
This confirms the last claim. □

Q Is T_pX a new object? **A**: It's a translation of the classical T_pX!

Proposition 1: Let $X \subseteq \mathbb{P}^n$ be a projective variety an $p \in X$. Fix $i \in \{0, \dots, n\}$ with $p \in U_i \simeq \mathbb{A}^n$ then
$$\underset{U_i \simeq \mathbb{K}^n}{(T_pX) \cap U_i} = \underset{\mathbb{K}^n}{T_p(X \cap U_i)} + p$$

Proof: Without loss of generality, we assume $i=0$ & set $p = (1, p_1, \dots, p_n) \in U_0 \cap X$
$$I(X \cap U_i) = \langle f(1, x_1, \dots, x_n) : f \in I(X) \text{ homogeneous} \rangle$$
$$= \langle f_1(1, x_1, \dots, x_n), \dots, f_r(1, x_1, \dots x_n) \rangle$$
if $I(X) = \langle f_1, \dots, f_r \rangle$ where $f_1, \dots, f_r$ are homogeneous.

- If $g(x_1, \dots, x_n) := f(1, x_1, \dots, x_n)$ then $\frac{\partial g}{\partial x_i}(p_1, \dots, p_n) = \frac{\partial f}{\partial x_i}(1, p_1, \dots, p_n) \;\forall i = 1, \dots n$
- Euler's identity applied to $f \in I(X)$ homogeneous & $\underline{x} = p$ yields:
$$\frac{\partial f}{\partial x_0}(1, p_1 \dots p_n) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(1, p_1, \dots p_n)\, p_i = 0$$

Therefore: $\frac{\partial f}{\partial x_0}(1, p_1, \dots, p_n) = -\sum_{i=1}^{n} p_i \frac{\partial f}{\partial x_i}(1, p_1 \dots p_n)$ for all $f \in I(X)$ homogeneous

Thus $\frac{\partial f}{\partial x_0}(1, p_1, \dots p_n) \cdot 1 + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(1, p_1, \dots p_n)\, x_i = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(p_1, \dots p_n)(x_i - p_i)$

Thus, $x \in T_p(X) \cap U_i \iff (x_1 - p_1, \dots, x_n - p_n) \in T_p(X \cap U_i) \subseteq \mathbb{K}^n = T_p(U_i)$ □

Corollary 1: Fix two projective varieties $X \subseteq \mathbb{P}^m$, $Y \subseteq \mathbb{P}^n$ & a rational map $F : X \dashrightarrow Y$ given by $n+1$ homogeneous polynomials $\{F_0, \ldots, F_n\}$ of the same degree. If $p \in X \cap \operatorname{Dom} F$ then F induces a linear map $dF_p \; T_pX \longrightarrow T_{F(p)}Y$. Furthermore,

(1) The map depends solely on F & not on the polynomials representing it

(2) If F is birational & F^{-1} is regular at $F(p)$, then dF_p is an isomorphism.

(3) The assignment is functorial.

Proof: The construction of dF_p follows from the affine case combined with Prop. 4.

More precisely $dF_p : T_pX \longrightarrow \mathbb{P}^n$

$$\underline{v} \longmapsto \left(\sum_{i=0}^{m} v_i \frac{\partial F_j}{\partial x_i}(p) \right)_{j=0}^{n}$$

If $I(X) = \langle g_1, \ldots, g_r \rangle$ with $g_1, \ldots, g_r$ homogeneous, then

$$\underline{v} \in T_pX \iff \sum_{j=0}^{m} v_j \frac{\partial g_i}{\partial x_j}(p) = 0 \quad \forall\, i = 1, \ldots, r \iff \sum_{j=0}^{m} v_j \frac{\partial g}{\partial x_j}(p) = 0$$

$\forall g \in I(X)$ homog.

If $I(Y) = \langle h_1, \ldots, h_s \rangle$ with $h_1, \ldots, h_s$ homogeneous, then

$$\underline{w} \in T_{F(p)}Y \iff \sum_{j=0}^{n} w_j \frac{\partial h_k}{\partial y_j}(F(p)) = 0 \quad \forall\, k = 1, \ldots s$$

- Since $F(X) \subseteq Y$, then $h_k(F_1(x), \ldots, F_K(x)) = 0 \quad \forall\, x \in V(\langle g_1, \ldots, g_r \rangle)$
- To show $dF_p(T_pX) \subseteq T_{F(p)}Y$ we only need to confirm

$$0 \overset{?}{=} \sum_{j=0}^{n} \underbrace{\left(\sum_{i=0}^{m} v_i \frac{\partial F_j}{\partial x_i}(p) \right)}_{w_j = dF_p(v)_j} \frac{\partial h_k}{\partial y_j}(F(p)) = \sum_{i=0}^{m} v_i \underbrace{\left(\sum_{j=0}^{n} \frac{\partial h_k}{\partial y_j}(F(p)) \frac{\partial F_j}{\partial x_i}(p) \right)}_{\frac{\partial (h_k \circ F)}{\partial x_i}(p)}$$

Since $h_k \circ F \in I(X)$ is homogeneous, the result follows because $\underline{v} \in T_pX$.

(1) Pick another tuple $(F_0' : \ldots : F_n')$ representing F. Thus $F_i' - F_i$ vanishes on V whenever defined. Thus $F_j' - F_j = \frac{P_j}{Q_j}$ where $P_j \in I(X)$ & $Q_j \in \mathbb{K}[X]$ with $Q_j(p) \neq 0$.

Thus

$$\frac{\partial (F_j' - F_j)}{\partial x_i}(p) = \left(\frac{\partial P_j}{\partial x_i}(p)\, Q_j(p) - \overset{=0}{P_j(p)} \frac{\partial Q_j}{\partial x_i}(p) \right) \Big/ Q_j^2(p)$$

$$= \frac{1}{Q_j(p)} \frac{\partial P_j}{\partial x_i}(p)$$

If $\underline{v} \in T_p X$ we have $\frac{1}{Q_j(p)} \sum_{i=0}^{m} v_i \frac{\partial P_j}{\partial x_i}(p) = \frac{1}{Q_j(p)} \cdot 0 = 0$, thus:

$\sum_{i=0}^{m} v_i \frac{\partial F'_j}{\partial x_i}(p) = \sum_{i=0}^{m} v_i \frac{\partial F_j}{\partial x_i}(p)$ ie dF_p is independent of the choice of tuple.

(3) The functoriality statement follows from the affine case ($d_p \psi \circ \varphi = d_{\varphi(p)} \psi \circ d_p \varphi$)

(2) is a consequence of the functoriality & the fact that the construction is local

$T_p U = T_p X$ & $T_{F(p)} V = T_{F(p)} Y$ if $U \underset{F^{-1}_{|V}}{\overset{F_{|U}}{\rightleftarrows}} V$

§2 Smooth Algebraic Varieties:

Fix X abstract algebraic variety over $\overline{\mathbb{K}} = \mathbb{K}$ & pick $p \in X$.

Definition: $\dim_p X := \dim \mathcal{O}_{X,p} = \operatorname{codim}_X p$.

Lemma 2: $\dim_p X = \max_{1 \leq i \leq s} \{ \dim_p X_i \mid p \in X_i \}$ if $X = X_1 \cup \dots \cup X_s$ is the irreducible decomposition of X

Proof: We may assume X is affine by working with an open affine U in X containing p. By construction, $U = (X_1 \cap U) \cup \dots \cup (X_s \cap U)$ is the irred decomp of X, extended possibly by $\emptyset$'s.

- If $\mathfrak{P}_i = I(X_i)$ $\quad p \in X_i \iff m_p \supseteq \underbrace{\mathfrak{P}_i \mathcal{O}_{X,p}}_{\text{prime ideal}} = \mathfrak{P}_i \left(\frac{\mathbb{K}[x]}{I(X)} \right)_{m_p}$

Since $\mathcal{O}_{X,p}$ has finite dimension, a maximal chain realizing $\dim \mathcal{O}_{X,p}$ corresponds to a maximal chain starting at a minimal prime of $\mathcal{O}_{X,p}$ & ending at m_p. Since minimal primes correspond to irred components of X, the result follows. □

Theorem 1: $\dim_{\mathbb{K}} T_p X \geq \dim_p X$.

Thus, a jump in dimension will detect singularities

Definition: A point $p \in X$ is nonsingular (or regular, or smooth) if

$$\dim_{\mathbb{K}} T_p X = \dim_p X$$

Otherwise, we say p is singular.

The variety X is nonsingular (or smooth) if all its points are smooth.

To prove Theorem 2, we show a more general statement:

Proposition 2: For every local ring $(R, \mathfrak{m})$ that is the localization of a $\mathbb{K}$-algebra of finite-type at a prime ideal, we have

$$\dim R \leq \dim_{\mathbb{K}} \mathfrak{m}/\mathfrak{m}^2$$

where $\mathbb{K} = \overline{\mathbb{K}} = R/\mathfrak{m}$

Proof: Write $R = A_{\mathfrak{p}}$ where $A = \mathbb{K}[x_1, \ldots x_n]/I$ & $\mathfrak{p} \subseteq A$ is a prime ideal

Claim 1: We can assume A is reduced (ie A has no nilpotents).

Pf/ Since localization is exact, we have

$$R_{red} := R/\sqrt{0} \simeq (A_{red})_{\overline{\mathfrak{p}}}$$

where $A_{red} := A/\sqrt{0}$ & $\overline{\mathfrak{p}} = \pi(\mathfrak{p}) \subseteq A_{red}$ under the projection $\pi: A \to A_{red}$

By construction, $\dim R_{red} = \dim R$ (any prime in R contains $\sqrt{0} \subseteq R$)

& $\mathfrak{m}_{red} = \overline{\mathfrak{p}}(A_{red})_{\overline{\mathfrak{p}}}$. Furthermore, exactness gives

$$R_{red}/\mathfrak{m}_{red} = \frac{(A_{red})_{\overline{\mathfrak{p}}}}{\overline{\mathfrak{p}}(A_{red})_{\overline{\mathfrak{p}}}} \underset{\uparrow \text{ exactness}}{=} \left(\frac{A_{\mathfrak{p}}}{\mathfrak{p}A_{\mathfrak{p}}}\right)_{red} = \mathbb{K}_{red} = \mathbb{K} = R/\mathfrak{m}$$

In turn $\mathfrak{m}_{red}^2 = (\mathfrak{m}^2 + \sqrt{0})/\sqrt{0}$ & $\mathfrak{m}_{red} = \mathfrak{m}/\sqrt{0}$

$$\dim_{\mathbb{K}} \mathfrak{m}_{red}/\mathfrak{m}_{red}^2 = \dim_{\mathbb{K}} \mathfrak{m}/(\mathfrak{m}^2 + \sqrt{0}) \leq \dim_{\mathbb{K}} \mathfrak{m}/\mathfrak{m}^2.$$

Thus, if the statement is true for $(R_{red}, \mathfrak{m}_{red})$, it is true for $(R, \mathfrak{m})$.

Claim 2: The statement is true for reduced local rings.

Pf/ Since R is reduced, we can write $R = A_{\mathfrak{p}}$ where $A = \mathbb{K}[x_1, \ldots x_n]/I$ is reduced.
In particular $A = \mathbb{K}[X]$ for $X = V(I) \subseteq \mathbb{A}^n$ affine variety by the Nullstellensatz.

Write $r = \dim_{\mathbb{K}} \mathfrak{m}/\mathfrak{m}^2$ & let $\{m_1, \ldots, m_r\}$ be a set of generators of $\mathfrak{m}$ in R

Set $m_i = \frac{a_i}{f_i}$ $a_i \in A$, $f_i \in A \smallsetminus \mathfrak{p}$. Write $\boxed{f = \prod_{i=1}^{r} f_i \in A \smallsetminus \mathfrak{p}.}$

- Since R is reduced, f is not nilpotent. Thus, $\{1, f, f^2, \dots\} \subseteq A$ is multiplicatively closed.
- By construction, $\mathfrak{P}A_f$ is generated by r elements $m_1', \dots, m_r'$, where $m_i' := (a_i \prod_{j \neq i} f_i)/f \quad \forall i = 1, \dots r$.
- Since $(A_f)_{\mathfrak{P}A_f} = A_{\mathfrak{P}}$, we can replace A by A_f which is a also reduced

$\left[\left(\frac{a}{f^r}\right)^m = 0 \text{ in } A_f \iff \exists k \geq 0 \text{ st } f^k a^n = 0 \right.$. Thus $f^{kn} a^n = (f^k a)^n = 0$

$\underset{A \text{ reduced}}{\implies} f^k a = 0 \implies \left. \frac{a}{f^r} = 0 \text{ in } A_f \right]$

Thus, $R = (A_f)_{\mathfrak{P}A_f}$ & $\dim R = \operatorname{codim}_{A_f} \mathfrak{P}A_f$. Note: $A_f = \mathbb{K}[D_X(f)]$

- $\mathfrak{P}A_f$ is the defining ideal of a variety in the quasi-affine variety $D_X(f)$

$$\boxed{Y := V_{D(f)}(m_1', \dots, m_r') \subseteq D_X(f) \subseteq X \subseteq \mathbb{A}^n}$$

Since $\mathfrak{P}A_f$ is prime, Y is irreducible. By Corollary 3 § 34.1, we have: $\operatorname{codim}_{D_X(f)} Y \leq r$, so $\dim R = \operatorname{codim}_{D_X(f)} Y \leq r$. $\square$