Lecture XXXVIII: Regular local rings & Smoothness of Varieties

Last Time: We defined smooth / singular points of varieties over $\overline{\mathbb{K}} = \mathbb{K}$.

Proposition: For every local ring (R, m) that is the localization of a $\mathbb{K}$-algebra of finite-type at a prime ideal, we have

$$\dim R \leq \dim_{\mathbb{K}} m/m^2$$

where $\mathbb{K} = \overline{\mathbb{K}} = R/m$

Theorem: Given X algebraic variety over $\overline{\mathbb{K}} = \mathbb{K}$ & $p \in X$ we have

$$\dim T_p X \geqslant \dim_p X := \dim \mathcal{O}_{X,p} = \max \{ \dim X_i : p \in X_i \}$$

if $X = X_1 \cup \cdots \cup X_s$ is the irreducible decomp of X

Definition: $p \in X$ is a smooth / non-singular / regular point $\Leftrightarrow$ $\dim T_p X = \dim_p X$

Q: What is the nature of a generic point of X?

§1 Regular local rings:

Remark: The core of the proof of Proposition 2 §37.2 is Krull's Principal Ideal Thm. Then, the statement is also true for Noetherian local rings.

Corollary 1: Given a Noetherian local ring (R, m) we have:

$$\dim R \leq \dim_{R/m} m/m^2$$

In particular, R has finite Krull dimension

Furthermore, equality holds if, and only if m is generated by a regular sequence (ie $\{f_1, \ldots, f_r\}$ st
- f_1 is a non-zero divisor in R
- $\forall i = 1, \ldots, r-1$ $\overline{f_{i+1}}$ is a non-zero divisor in $R/\langle f_1, \ldots, f_i \rangle$)

Definition: We say a Noetherian local ring (R, m) is regular if $\dim R = \dim_{R/m} m/m^2$

For such rings, a regular system of parameters for m is a minimal set of generators of m (hence, of size $= \dim R$)

Theorem 1: A Noetherian regular local ring is a domain.

• We will show a slightly weaker result, and then use it to prove Theorem 1. We'll need the following definition:

Definition: Given (R, m) Noetherian local ring, the associate graded ring of R:

$$S := gr_m R = \bigoplus_{k \geq 0} m^k / m^{k+1}$$

Name: $S_k := m^k / m^{k+1}$ k^{th} graded piece

• $S_0 = m^0 / m = R/m = \mathbb{K}$ (residue field)

• S is a ring with multiplication between graded pieces defined as

$$\begin{array}{ccc} m^k / m^{k+1} \times m^\ell / m^{\ell+1} & \longrightarrow & m^{k+\ell} / m^{k+\ell+1} \\ (\bar{a}, \bar{b}) & \longmapsto & \overline{ab} \end{array}$$

(extended linearly to $S \times S \longrightarrow S$)

In particular, S_k is a $\mathbb{K}$-vector space $\forall k$

Remark: The construction is more general. For any A Noetherian ring & $I \subseteq A$ ideal, the associated graded $gr_I R = \bigoplus_n I^n / I^{n+1}$ is a Noetherian ring & an R/I-module.

Theorem 2: If (R, m) is regular Noetherian local ring, then $gr_m R$ is a domain. Furthermore, it is isomorphic to $\mathbb{K}[y_1, \ldots, y_d]$ where $d = \dim R$.

Proof: Set $d = \dim R$ & pick $a_1, \ldots, a_d$ generators for m. Since R is regular, then $\{a_1, \ldots, a_d\}$ is a regular sequence. Set $S = gr_m R$

By construction, $\{\bar{a}_1, \ldots, \bar{a}_d\} \in S_1$ is a $\mathbb{K}$-basis for S_1.

Define the graded k-algebra homomorphism

$$\begin{array}{ccc} \varphi : \mathbb{K}[x_1, \ldots, x_d] & \longrightarrow & S \\ x_i & \longmapsto & \bar{a}_i \end{array}$$

so $\varphi(P(\underline{x})) = P(\bar{a}_1, \ldots, \bar{a}_d)$

By construction, φ is surjective.

Claim 1: $\dim R = \dim gr_m R$ because R is a Noetherian local ring

Pf/ $\dim gr_m R = \dim R_m$ (see Exercise 13.8 in Eisenbud's Comm. Alg book)

Claim 2: φ is injective:

Pf/ φ is a surjective ring homomorphism between two rings of the same dimension, and the domain of φ is a domain.

A chain of prime ideals $\wp_0 \subsetneq \wp_1 \subsetneq \dots \subseteq \wp_s$ in $gr_m R$ of length length $d = \dim gr_m R$ yields a proper chain of prime ideals in R since R is surjective:

$$q_0 \subsetneq q_1 \subsetneq \dots \subsetneq q_d$$

If $q_0 \neq \{0\}$, we can extend this to a maximal chain in $\mathbb{K}[x_1,\dots x_d]$ of length $s+1$ by $0 \subsetneq q_0 \subsetneq q_1 \subsetneq \dots \subsetneq q_d$.

This cannot happen because $\dim \mathbb{K}[x_1,\dots x_d] = d$ by Theorem 1 §33.2.

Conclusion: $q_0 = \{0\}$ so $\wp_0 = \varphi(0) = \{0\}$ ie $q_0 = \operatorname{Ker}\varphi = \{0\}$.

Proposition 1: If (R, m) is Noetherian local & $gr_m R$ is a domain, then so is R

Proof: We set $S = gr_m R$ & assume it is a domain. We want to show R is also a domain. We argue by contradiction & pick $a, b \in R \smallsetminus \{0\}$ with $ab = 0$

By Krull's Intersection Theorem: $\bigcap m^n = \{0\}$ (because (R, m) is Noetherian & local). Thus, $\exists\, i, j$ with $a \in m^i \smallsetminus m^{i+1}$ & $b \in m^j \smallsetminus m^{j+1}$.

In particular $\bar{a}, \bar{b} \in S \smallsetminus \{0\}$ & $\overline{ab} = 0$ in S. This cannot happen since S is a domain. $\square$

Proposition 2: If p is a smooth point of an alg variety X over $\overline{\mathbb{K}} = \mathbb{K}$, then $\mathcal{O}_{X,p}$ is a domain. In particular, p lies on a unique irreducible component of X.

Proof: Write $R = \mathcal{O}_{X,p}$ & let $m = m_p \subseteq \mathcal{O}_{X,p}$ be the maximal ideal. Recall that $R/m \simeq \mathbb{K}$ since $\overline{\mathbb{K}} = \mathbb{K}$. By Corollary 1, R is a Noetherian regular local ring.

- By Theorem 1, $gr_m R$ is a domain.
- By Proposition 1, R is a domain.

Claim: $\mathcal{O}_{X,p}$ is a domain $\iff$ p lies in a unique irreducible component of X

Pf/. Components of X correspond to minimal primes of $\mathbb{K}[X] = \mathcal{O}_X(X)$.

. ______ containing p correspond to minimal primes of $\mathcal{O}_{X,p} = \mathbb{K}[X]_{m_p}$

Since $\mathcal{O}_{X,p}$ is always reduced, $\mathcal{O}_{X,p}$ has a unique minimal prime $\mathcal{P} \iff \mathcal{P} = \{0\}$, ie if & only if $\mathcal{O}_{X,p}$ is a domain. $\square$

From Proposition 2, we get an easy consequence

Corollary 2: Every smooth hypersurface in $\mathbb{P}^n$ is irreducible

§2. Smooth Locus of Algebraic Varieties:

Definition: Given an algebraic variety X over $\overline{\mathbb{K}} = \mathbb{K}$, we write

$$X_{sm} := \{p \in X : p \text{ is a smooth point}\}$$

Theorem 3: X_{sm} is a dense open subset of X.

To prove the theorem we will do it in 2 stages:

(1) Prove it for irreducibles (Proposition 3 below & Lemma 1 next time)

(2) Show if $X = X_1 \cup \cdots \cup X_m$ is an irred decomp, then $X_{sm} \cap X_i \cap X_j = \emptyset \ \forall i \neq j$

(ie, points in 2 irreducible components are always singular.) (Proposition 2 above)

Proof: For each i, set $X_i' := X_i \setminus \bigcup_{j \neq i} X_j$. By construction, X_i' is an irreducible variety (it is open in the irreducible variety X_i). By (1) $(X_i')_{sm}$ is dense open in X_i'

By (2) $X_{sm} = \bigcup_{i=1}^{m} (X_i')_{sm}$, so X_{sm} is open dense in $\bigcup_{i=1}^{m} X_i'$ (open in X)

But each X_i' is dense in X_i because $X_i' \subseteq X_i$ is non-empty open & X_i is irreducible.

Thus, X_{sm} is open dense in X. $\square$

Proposition 3: If X is an irreducible variety, X_{sm} is open in X

Proof: Write $X = U_1 \cup \cdots \cup U_r$ for an affine open cover.

Claim 1: If $X_{sm} \cap U_i = (U_i)_{sm}$ is open in U_i, then X_{sm} is open in X

Pf/ $(U_i)_{sm} \subseteq U_i$ is open & $U_i \subseteq X$ is open, so $(U_i)_{sm} \subseteq X$ is open

Thus $X_{sm} \subseteq X$ is open because $X_{sm} \cap U_i \subseteq U_i$ is ______ $\forall i$ $\square$

Claim 2: The statement is true for irreducible affine varieties

Pf/ Write $I(X) = \langle f_1, \dots, f_r \rangle \subseteq \mathbb{K}[x_1, \dots x_n]$ It is a prime ideal & $X = V(f_1, \dots, f_r)$ is irreducible. By Corollary 3 § 34.1 $\operatorname{codim}_{\mathbb{A}^n} X \leq r$

In turn, $\dim X \geq n-r$ by Theorem 1 § 33.2.

- Since X is irreducible, we know $\dim_p X := \dim \mathcal{O}_{X,p} = \dim X = d \quad \forall p \in X$.
- Next, we show $X_{sm} \subseteq X$ is open.

Set $\operatorname{Jac}(\underline{f}, q) := \operatorname{Jac}(f_1, \dots f_r, q) := \left(\frac{\partial f_i}{\partial x_j}(q) \right)_{i,j} \in \mathbb{K}^{r \times n}$ for every $q \in X$.

By Proposition 1 § 36.1 $T_p X = \operatorname{Ker}\left(\operatorname{Jac}(\underline{f}, p) \right)$ has $\dim = n - \operatorname{rk}(\operatorname{Jac}(\underline{f}, q))$

Since $\dim T_p X \geq d$ we know $\operatorname{rk}(\underline{f}, q)) \leq n-d \quad \forall q \in X$. $\geq n-r$

Thus, all $(n-d+1) \times (n-d+1)$ minors of $\operatorname{Jac}(\underline{f}, q)$ vanish whenever $q \in X$.

Then $p \in X$ is non-singular if and only if $\dim T_p X = d$, ie $\operatorname{rk}(\operatorname{Jac}(\underline{f}, p)) = n-d$

Thus $X_{sm} = X \cap \bigcup_{\substack{|I| = n-d \\ |I'| = n-d}} D(\operatorname{minor}_{I,I'})$ where $I \subseteq \{1, \dots, r\}$ $(r \geq n-d)$, $I' \subseteq \{1, \dots, n\}$ index the rows & columns of the corresponding $(n-d) \times (n-d)$ submatrix $J_{I,I'}$ of $\operatorname{Jac}(\underline{f}, p)$ giving the minor $\operatorname{minor}_{I,I'} = \det J_{I,I'}$. We conclude X_{sm} is open in X. $\square$

- **Next Time:** we'll see X_{sm} is non-empty.