Lecture XXXIX: Smoothness & Blow-ups of affine varieties

§1 Smooth points on varieties:

Definition: Given an algebraic variety X over $\overline{\mathbb{K}}=\mathbb{K}$, we write

$$X_{sm} := \{p \in X : p \text{ is a smooth point}\}$$

Last time we proved the following result:

Proposition 1: X_{sm} is an open subset of X.

Proof sketch: We reduced to the case where X is affine and irreducible with defining ideal $I(X)=\langle f_1,\ldots,f_r\rangle \subseteq \mathbb{K}[x_1,\ldots,x_n]$. We showed:

$$X_{sm} = \{p\in X \mid Jac(\underline{f},p) = \left(\frac{\partial f_i}{\partial x_j}\right)_{i,j} \text{ has maximal rank} = n-d\}$$

for $d=\dim X$ ($=\dim_p X$ for all $p\in X$). (Recall: $n-d\leq r$)

- We know $rk(Jac(\underline{f},p)) \leq n-d$ if $p\in X$ because $\dim T_pX \geq \dim X$.
- So the maximal rank condition becomes the non-vanishing of at least one of the $(n-d)\times(n-d)$ minors of $Jac(\underline{f},p)$ with $p\in X$. This is an open condition. □

- As a consequence of the proof, we get:

Corollary 1 (Affine Jacobian criterion)

Fix $X\subseteq \mathbb{A}^n$ an affine variety over $\overline{\mathbb{K}}=\mathbb{K}$ & $p\in X$. Write $I(X)=\langle f_1,\ldots,f_r\rangle$

(1) X is smooth at p if and only if the rank of the $r\times n$ Jacobian matrix

$$Jac(\underline{f},p) = \left(\frac{\partial f_i}{\partial x_j}(p)\right)_{i,j}$$

is at least $n-\dim_p X$. In fact, it must agree with $n-\dim_p X$.

(2) If $r\leq n$ & $rk\, Jac(\underline{f},p)=r$ then, X is smooth at p & $\dim_p X = n-r$.

Proof: (1) The rank of the Jacobian matrix is $n-\dim T_pX \leq n-\dim_p X$ by the proof of Proposition 1.

Thus $n-\dim T_pX \geq n-\dim_p X \iff \dim_p X = \dim T_p X$.

(2) By Corollary 3 §34.1, $\operatorname{codim}_X X_i \leq r$ $\forall X_i$ component of X. In particular, by Lemma 2 §37.2 $\dim_p X \geq n-r = \dim T_pX$

thus by Theorem 1 §37.1, $\dim_p X = \dim T_pX$ so p is a smooth point of X & $\dim_p X = n-r$. □

Using Proposition 1 §37.1, we get a projective version of the same criterion

Corollary 2 (Projective Jacobian criterion)

Fix $X \subseteq \mathbb{P}^n$ a projective variety over $\overline{\mathbb{K}} = \mathbb{K}$ & $p \in X$. Write $I(X) = \langle f_1, \dots, f_r \rangle$ where $f_1, \dots, f_r$ are homogeneous

Then X is smooth at p if and only if the rank of the $r \times (n+1)$ Jacobian matrix

$$\left(\frac{\partial f_i}{\partial x_j}(p) \right)_{i,j}$$

is at least $n - \dim_p X$. In fact, it must agree with $n - \dim_p X$.

- Not only $X_{sm} \subseteq X$ is open, but we have:

Theorem 1: $X_{sm} \subseteq X$ is not empty. Furthermore, it is dense in X.

Proof: It is enough to show this for X an irreducible quasi-affine variety

Claim 1: We can assume that X is affine

Pf/ Write an open affine cover of the variety X: $X = U_1 \cup \cdots \cup U_s$.

Then $X_{sm} \cap U_i = (U_i)_{sm}$ $\forall i$

- If $(U_i)_{sm} \subseteq U_i$ is non-empty, then so is X_{sm}.
- If $(U_i)_{sm}$ is dense in U_i, then $X_{sm} = \bigcup_i (X_{sm} \cap U_i)$ is dense in X.

Claim 2: We can assume that X is an irreducible quasi-affine variety

Pf/ By Claim 1 we can assume X is affine. If $X = X_1 \cup \cdots \cup X_s$ is irreducible decomposition, then the condition "$\mathcal{O}_{X,p}$ is a domain if $p \in X_{sm}$" (Proposition 2 §38.1) forces

$$X_{sm} = \bigcup_{j=1}^{s} \underbrace{\left(X_j \smallsetminus \bigcup_{i \neq j} X_i \right)}_{= W_j}{}_{sm} \quad \& \quad W_j \subseteq X_j \text{ is open in } X_j \text{ irred affine}$$

so W_j is an irreducible quasi-affine variety

Since $(W_j)_{sm} \subseteq W_j$ is open by proposition, then $(W_j)_{sm} \neq \emptyset$ would say $(W_j)_{sm}$ is dense in X_j. Thus $X_{sm} \neq \emptyset$ & X_{sm} is dense in X. □

- Assume X is irreducible quasi-affine. To show $X_{sm} \neq \emptyset$, it is enough to show it for any Y irreducible birational to X.

By Lemma 1 below, we can pick such Y to be an irreducible hypersurface in $\mathbb{A}^{d+1}$

ie $Y = V(f)$ for some $f \in \mathbb{K}[y_1, \ldots, y_d]$ irreducible

In this setting $\mathrm{Jac}(f, q)$ has size $1 \times d$ & its rank is 1 unless

$$f(q) = \frac{\partial f}{\partial y_1}(q) = \cdots = \frac{\partial f}{\partial y_n}(q)$$

Thus $Y_{sm} = \phi \iff \frac{\partial f}{\partial y_i} \in \langle f \rangle \quad \forall\, i = 1, \ldots, d$

But if $\deg_{y_i} f = d_i$, then $\frac{\partial f}{\partial y_i} = 0$ or $\deg \frac{\partial f}{\partial y_i} = d_i - 1$.

Therefore $\frac{\partial f}{\partial y_i} \in \langle f \rangle \, \forall i \iff \frac{\partial f}{\partial y_i} = 0 \quad \forall i$

This can only happen if $\mathrm{char}\, \mathbb{K} = p > 0$ & $f \in \mathbb{K}[y_1^p, \ldots, y_n^p]$

Since $\overline{\mathbb{K}} = \mathbb{K}$, then $\mathbb{K}$ is perfect & $f \in \mathbb{K}[y_1^p, \ldots, y_n^p]$ is equivalent to the condition that $f = g^p$ for some $g \in \mathbb{K}[y_1, \ldots, y_n]$. This cannot happen if f is irreducible □

<u>Lemma 1</u>: Every irreducible affine variety over $\overline{\mathbb{K}} = \mathbb{K}$ is birational to an irreducible hypersurface in an affine variety (ie $= V(f)$)

<u>Proof:</u> Fix X irreducible affine variety. Consider the function field $k = \mathbb{K}(X)$, which is a field extension of $\overline{\mathbb{K}} = \mathbb{K}$. Furthermore $\mathrm{Tr}\deg_{\mathbb{K}} k = \dim X = d < \infty$.

<u>Note</u>: k is finitely generated over $\mathbb{K}$ $\quad (X \subseteq \mathbb{A}^n \Rightarrow k = \mathrm{Quot}(\mathbb{K}[X]) = \mathbb{K}(\bar{x}_1, \ldots \bar{x}_n))$

<u>Claim</u>: $\exists$ a transcendental basis $\{y_1, \ldots, y_d\}$ of $k \mid \mathbb{K}$ such that k is separable over $\mathbb{K}(y_1, \ldots y_d)$ & finite (i.e, k is separably generated over $\mathbb{K}$)

Pf/ See Lemma 2 below

Pick such basis. By the Primitive Element Theorem $\exists\, u \in k$ s.t. $k = \mathbb{K}(y_1, \ldots, y_d, u)$.

Write $f = \min(u, \mathbb{K}(y_1 \cdots y_d)) \in \mathbb{K}(y_1, \ldots, y_d)[t]$. Then

$$k \cong \mathbb{K}(y_1, \ldots, y_d)[t] / \langle f \rangle$$

After multiplying u by a suitable non-zero element of $\mathbb{K}(y_1, \ldots, y^d)$, we can assume $f \in \mathbb{K}[y_1, \ldots, y_d, t]$ & f is irreducible. Then $Y = V(f) \subseteq \mathbb{A}^{d+1}$ satisfies

$$\mathbb{K}(Y) = \mathrm{Quot}(\mathbb{K}[y_1, \ldots, y_d, t]/(f)) = \mathrm{Quot}(\mathbb{K}(y_1 \cdots y_n)[t]/\langle f \rangle) \simeq k = \mathbb{K}(X)$$

Thus, X & Y are birational to each other by Corollary 3 §11.2. □

Lemma 2: If $\mathbb{K}$ is a perfect field (ie $\operatorname{char}\mathbb{K}=0$ or $\operatorname{char}\mathbb{K}=p>0$ & $\mathbb{K}=\mathbb{K}^p$), and $F|\mathbb{K}$ is a finitely generated extension, then there exists a transcendental basis $\{x_1,\dots x_d\}$ of $F|\mathbb{K}$ s.t. F is separable & finite over $\mathbb{K}(x_1,\dots x_d)$.

Proof. • If $\operatorname{char}\mathbb{K}=0$, there is nothing to do (all extensions are separable). Indeed, any transcendental basis $\{x_1,\dots x_d\}$ of $F|\mathbb{K}$ works because $F|\mathbb{K}(x_1,\dots x_d)$ is algebraic & finitely generated, then it must be finite as well.

• Next, assume $\operatorname{char}\mathbb{K}=p>0$ & write $F=\mathbb{K}(x_1,\dots,x_m)$. We can pick a transcendental basis for $F|\mathbb{K}$ among $\{x_1,\dots,x_m\}$. Up to reordering, we assume $\{x_1,\dots,x_d\}$ is such transcendental basis. Furthermore, assume $\{x_{d+1},\dots x_s\}$ are not separable over $L:=\mathbb{K}(x_1,\dots x_d)$ whereas $\{x_{s+1},\dots,x_m\}$ are. We procede by induction on s.

• If $s=0$, we are done.

• Otherwise, since x_{d+1} is not separable over L, $\exists$ $f\in L[t]$ irreducible s.t. $f\in L[t^p]$ & $f(x_{d+1})=0$.

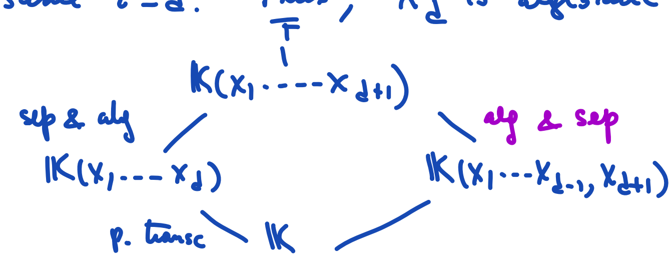
Pick $u\in\mathbb{K}(x_1,\dots,x_d)$ st $g=uf\in\mathbb{K}[x_1,\dots x_n,t^p]$ is irreducible & $g(x_{d+1})=0$

Claim: $\exists$ $i\leq d$ such that $\frac{\partial g}{\partial x_i}\neq 0$

Pf/ We argue by contradiction. If $\frac{\partial g}{\partial x_1}=\dots=\frac{\partial g}{\partial x_d}=0$, then $g\in\mathbb{K}[x_1^p,\dots x_d^p,t^p]$

Since $\mathbb{K}$ is perfect, $\mathbb{K}=\mathbb{K}^p$ hence $g=h^p$ with $h\in\mathbb{K}[x_1,\dots x_d,t]$. This cannot happen because g is irreducible. □

• After relabeling the variables, we assume $i=d$. Thus, x_d is algebraic & separable over $\mathbb{K}(x_1,\dots,x_{d-1},x_{d+1})$.



Thus $\{x_1,\dots,x_{d-1},x_{d+1}\}$ is a transcendental basis for $F|\mathbb{K}$.

• x_j is sep over $\mathbb{K}(x_1,\dots x_d)$ for $j\geq s+1$ $\Longrightarrow$ x_j is separable over $\mathbb{K}(x_1,\dots x_{d-1},x_{d+1})$

(because $F \mid \mathbb{K}(x_1, \ldots x_{d+1})$ is separable & x_d is sep. over $\mathbb{K}(x_1, \ldots x_{d-1}, x_{d+1})$.)

Thus, swapping x_d with x_{d+1} produces $\leq s-1-d$ unseparable variables. By inductive hypothesis, we can swap these elements with some of $\{x_1, \ldots x_{d-1}, x_{d+1}\}$ to produce the desired transcendental basis ∎

Now that we know $X_{sm} \subseteq X$ is open & dense, we can ask:

Q: Can we find Y variety birational to X, $X \overset{\varphi}{\dashrightarrow} Y$ with $X_{sm} \subseteq \mathrm{Domain}$ of X & $X_{sm} \xrightarrow{\sim} Y_{sm}$?

A: Blow-ups are going to help us achieve this if char $\mathbb{K} = 0$.

§2 Blow-ups of affine varieties over $\overline{\mathbb{K}} = \mathbb{K}$:

Throughout, we assume $X \subseteq \mathbb{A}^n$ is an affine variety & $\overline{\mathbb{K}} = \mathbb{K}$

Pick $f_1, \ldots, f_r \in \mathcal{O}(X)$ & set $U = X \setminus V(f_1, \ldots, f_r) \subseteq X$ (open).

- This defines a map

$$\underline{f} : U \longrightarrow \mathbb{P}^{r-1}$$
$$x \longmapsto [f_1(x) : \cdots : f_r(x)]$$

By construction, $\underline{f}$ is a regular morphism between varieties. (**Note:** U can be $\emptyset$)

- Consider the graph of $\underline{f}$:

$$\boxed{\Gamma_{\underline{f}} = \{(x, \underline{f}(x)) : x \in U\} \subseteq U \times \mathbb{P}^{r-1}}$$

By Proposition 2 (1) §25.2, $\Gamma_{\underline{f}}$ is closed in $U \times \mathbb{P}^{r-1}$, but not necessarily in $X \times \mathbb{P}^{r-1}$.

Definition: $\widetilde{X} := \overline{\Gamma_{\underline{f}}} \subseteq X \times \mathbb{P}^{r-1}$ is called the blow-up of X at $f_1, \ldots, f_r$.

Q: What are the defining equations of $\widetilde{X}$?

A: NOT easy! A partial list of equations will be given next time.

Remark: ① We have a natural map $\widetilde{X} = \overline{\Gamma_{\underline{f}}} \xrightarrow{p_1} X$ we call it the blow-up map. Usually, we write it as $\pi : \widetilde{X} \longrightarrow X$.

②
$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\pi} & X \\ \cup\!\text{I open} & & \cup\!\text{I open} \\ \Gamma_{\underline{f}} & \xrightarrow{\sim} & U \end{array}$$

. Furthermore, we can view $U \simeq \Gamma_{\underline{f}} \subseteq \widetilde{X}$ as an open set.

In particular, if X is irreducible, with $I(X) = \langle g_1, \dots, g_s \rangle$, of dimension d, then we can take $U = X_{sm} = X \setminus V(f_1, \dots f_r)$ where $\{f_1, \dots, f_r\}$ is the list of all $(n-d)\times(n-d)$ minors of $\mathrm{Jac}(\underline{g}, p)$. We have $X_{sm} \subseteq \tilde{X}$.

Furthermore, X_{sm} becomes dense in $\tilde{X}$ by the following result.

<u>Lemma 3</u>: If X is irreducible & $X \not\subseteq V(f_1, \dots, f_r)$, then $\tilde{X} \xrightarrow{\pi} X$ is a birational map with $\Gamma_f \xrightarrow[\sim]{\pi} U$ dense opens in $\tilde{X}$ & X, respectively.

<u>Proof</u>: Since $X \not\subseteq V(f_1, \dots, f_r)$, then $U = X \setminus V(f_1, \dots f_r)$ is open & non-empty in X. But X is irreducible, hence U is dense in X.

In this case, $U \simeq \Gamma_{\underline{f}} \subseteq \tilde{X}$ is also open. Since $\Gamma_{\underline{f}}$ is dense in $\overline{\Gamma_{\underline{f}}} = \tilde{X}$ by definition, so is U. We conclude $\pi: \tilde{X} \longrightarrow X$ is a birational map, restricting to an isomorphism between the dense opens Γ_f & U, respectively. □

- Next time, we'll discuss an important result, namely:

<u>Theorem</u>: The construction of $\tilde{X}$ depends solely on $\langle f_1, \dots, f_r \rangle$ (up to isomorphism).