# Lecture XL: Blow-ups of affine varieties II

**Recall**: $X \subseteq \mathbb{A}^n$ affine variety over $\overline{\mathbb{K}}=\mathbb{K}$ & $f_1,\dots,f_r \in \mathcal{O}(X)$, then

$$\underline{f} : U = X \setminus V(f_1,\dots,f_r) \longrightarrow \mathbb{P}^{r-1} \quad \text{is a regular morphism}$$

$$\Gamma_{\underline{f}} = \{(x,\underline{f}(x)) : x \in U\} \text{ is closed in } U \times \mathbb{P}^{r-1}$$

**Definition**: $\widetilde{X} = \overline{\Gamma_{\underline{f}}} \subseteq X \times \mathbb{P}^{r-1}$ is the blow-up of $X$ along $f_1,\dots,f_r$.

- $\widetilde{X}$ comes with a natural map $\pi : \widetilde{X} \longrightarrow X$   **Name**: Blow-up map.

$$\begin{tikzcd} \widetilde{X} \arrow[rr, "\pi"] \arrow[dr, hook] & \circlearrowleft & X \\ & X \times \mathbb{P}^{r-1} \arrow[ur, "p_1"'] & \end{tikzcd}$$

- $\pi|_{\Gamma_{\underline{f}}}$ is an iso between $\Gamma_{\underline{f}}$ & $U$, so we have $U \hookrightarrow \widetilde{X}$.

$\Gamma_{\underline{f}} \simeq U$ under $\pi$ is dense in $\widetilde{X}$ if non-empty. This happen if $X$ is irreducible & $X \not\subseteq V(f_1,\dots,f_r)$

## §1 More basics on blow-ups:

Set $E = \widetilde{X} \setminus U = \pi^{-1}(V(f_1,\dots f_r)) \subseteq X$ is closed

**Definition**: $E$ is called the exceptional set of the blow-up.

⚠ Usually, $\pi$ is not an ismorphism on $E$.

- If $X$ is irreducible & $U \neq \emptyset$, then $U \hookrightarrow \widetilde{X}$ is open & dense.

**Q1**: What happens if $X$ is reducible?

**Q2**: What ——— we take $Y \subseteq X$ subvariety? How are $\widetilde{Y}$ & $\widetilde{X}$ related?

Answering Q2 will help us with Q1 because $Y$ = irred comp of $X \subseteq X$ is a subvariety!

**Lemma 1**: Fix $Y \subseteq X$ a subvariety, $f_1,\dots,f_r \in \mathcal{O}(X)$. We view $f_1,\dots,f_r \in \mathcal{O}(Y)$, & set $V = Y \setminus V(f_1,\dots f_r)$. Then, the blow-ups $\widetilde{X}$ & $\widetilde{Y}$ of $X$ & $Y$ at $f_1,\dots,f_r$ satisfy

(1) $\widetilde{Y} \subseteq \widetilde{X}$ is closed

(2) $\widetilde{Y} = \overline{\Gamma_{\underline{f}} \cap (Y \times \mathbb{P}^{r-1})} = \overline{\pi^{-1}(U) \cap (Y \times \mathbb{P}^{r-1})} = \overline{\pi^{-1}(Y \setminus V(f_1,\dots f_r))}$

**Definition**:
- $\widetilde{Y}$ is called the strict or proper transform of $Y$ in the blow-up of $X$
- $\pi^{-1}(Y)$ ——— total transform   ( $\widetilde{Y} \neq \pi^{-1}(Y)$ iff $V \neq Y$ )

**Proof:** By construction: $\tilde{Y} \subseteq Y \times \mathbb{P}^{r-1} \subseteq X \times \mathbb{P}^{r-1}$

$$\tilde{Y} = \overline{\Gamma_{\underline{f}|_Y}}^{Y\times\mathbb{P}^{r-1}} = \overline{\Gamma_f \cap (Y \times \mathbb{P}^{r-1})}^{X\times\mathbb{P}^{r-1}} \subseteq \tilde{X}$$

closed

(1). • Since $Y$ is closed in $X$, then $Y\times\mathbb{P}^{r-1}$ is closed in $X\times\mathbb{P}^{r-1}$

• Since $\tilde{Y} = \overline{\Gamma_{\underline{f}|_Y}}^{Y\times\mathbb{P}^{r-1}}$ then $\tilde{Y}$ is closed in $Y\times\mathbb{P}^{r-1}$ & $\overline{\Gamma_{\underline{f}|_Y}}^{Y\times\mathbb{P}^{r-1}} = \overline{\Gamma_{\underline{f}|_Y}}^{X\times\mathbb{P}^{r-1}}$

**Conclude:** $\tilde{Y}$ is closed in $\tilde{X}$

(2) $\tilde{Y} = \overline{\Gamma_{\underline{f}|_Y}}^{Y\times\mathbb{P}^{r-1}} = \overline{\Gamma_{f|_Y}} = \overline{\Gamma_f \cap Y\times\mathbb{P}^{r-1}} = \overline{\pi^{-1}(Y\cap U)} \subseteq \tilde{X}$ since $\Gamma_{\underline{f}} \simeq \pi^{-1}(U) \subseteq \tilde{X}$

**Corollary 1:** If $X = X_1 \cup \cdots \cup X_r$ irred. decomp., then $\tilde{X} = \tilde{X}_1 \cup \cdots \cup \tilde{X}_r$

**Proof:** Taking closure commutes with taking finite unions, so

$$\begin{aligned}\tilde{X}_1 \cup \cdots \cup \tilde{X}_r &= \overline{\pi^{-1}(X_1\cap U)} \cup \cdots \cup \overline{\pi^{-1}(X_r\cap U)} \\ &= \overline{\pi^{-1}(X_1\cap U) \cup \cdots \cup \pi^{-1}(X_r\cap U)} = \overline{\pi^{-1}((X_1\cap U)\cup\cdots\cup(X_r\cap U))} \\ &= \overline{\pi^{-1}((X_1\cup\cdots\cup X_r)\cap U)} = \overline{\pi^{-1}(X\cap U)} = \overline{\pi^{-1}(U)} = \tilde{X}\end{aligned}$$

**Corollary 2:** $\dim \tilde{X} = \dim X \quad \forall X$

**Proof:** $\dim X = \max_i \dim X_i \underset{(*)}{=} \max_i \dim \tilde{X}_i = \tilde{X}$    ((*) $\pi_i : \tilde{X}_i \to X_i$ is birational)

We know that $\tilde{X} \subseteq X \times \mathbb{P}^{r-1}$ is closed.

**Q:** What are the defining equations? **A:** Hard, but we get an easy partial list:

**Proposition 1:** The blow-up of $X \subseteq \mathbb{A}^n$ affine at $f_1, \ldots, f_r \in \mathcal{O}(X)$ satisfies

$$\tilde{X} \subseteq \{(x,y) \in X \times \mathbb{P}^{r-1} : y_i f_j(x) = y_j f_i(x) \quad \forall i,j = 1,\ldots,r\}$$

Thus, $\tilde{X} \subseteq V(\, I(X)\, \mathbb{K}[\underline{x},\underline{y}] + \langle y_i f_j(\underline{x}) - y_j f_i(x) \quad i,j=1,\ldots,r \rangle)$

$\underline{x} = (x_1, \ldots, x_n)$

$\underline{y} = (y_1, \ldots, y_r)$

↳ can be replaced by polynomials in this ideal, homogeneous in $y$ variables

**Proof:** Any $(x,y) \in U \times \mathbb{P}^{r-1}$ satisfies $(y_1, \ldots, y_r) = (f_1(x) : \cdots : f_r(x))$

Equivalently $y_i f_j(x) - y_j f_i(x) = 0$ & $\underline{x} \in U$.

Since $U = X \setminus V(f_1, \ldots f_r)$ is open in $X$, & $\Gamma_f \simeq U$ under the projection $p_1$,

these equations will hold automatically on $\tilde{X} = \overline{\Gamma_f} \subseteq X \times \mathbb{P}^{r-1}$

## §2. Examples:

**Example 1:** Take $r=1$. Then $\Gamma_f = \{(x, [f_1(x)]) \;\; x \in X \setminus V(f_1)\} = X \setminus V(f_1)$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \overset{\shortparallel}{[1]}$

$$\text{So} \quad \tilde{X} = \overline{X \setminus V(f_1)} = \overline{U}^{X}$$

Assuming $X$ is irreducible, there are 2 options:

① $f_1 = 0$ in $\mathcal{O}(X)$, so $U = X \setminus V(f_1) = \emptyset$. Thus $\tilde{X} = \emptyset$

② $f_1 \neq 0$ in $\mathcal{O}(X)$, so $U \neq \emptyset$ is dense. Hence $\tilde{X} \simeq X$.

Our next example gives an instance where equality in Proposition 1 holds.

**Example 2:** Take $X = \mathbb{A}^n$ & $\underline{f} = (x_1, \ldots, x_n)$

$U = \mathbb{A}^n \setminus \{(0, 0, \ldots, 0)\}$

$$\tilde{\mathbb{A}}^n = \overline{\{(\underline{x}, \underline{x}) \in (\mathbb{A}^n \setminus \{0\}) \times \mathbb{P}^{n-1}\}} \overset{\text{Prop 1}}{\subseteq} \{(\underline{x}, [\underline{y}]) : x_i y_j - x_j y_i \;\; \forall i,j\}$$

**Claim:** $\tilde{\mathbb{A}}^n = \{(\underline{x}, [\underline{y}]) \in \mathbb{A}^n \times \mathbb{P}^{n-1} : x_i y_j - x_j y_i \;\; \forall i,j\} =: Y$

Pf/ $Y = \{(\underline{x}, [\underline{x}]) \in (\mathbb{A}^n \setminus \{0\}) \times \mathbb{P}^{n-1}\} \cup (\{0\} \times \mathbb{P}^{n-1})$

Fix $i \in \{1, \ldots, n\}$ & consider $\boxed{V_i = \mathbb{A}^n \setminus \{x_i = 0\} \subseteq \mathbb{A}^n}$ open, $V_i$ irreducible quasi-affine

The equations in $Y$ yield $Y \cap (V_i \times \mathbb{P}^{n-1}) = \{(\underline{x}, [x]) \in V_i \times U_{i-1}\} =: Y_i$

- $Y_i \simeq V_i$, $\dim V_i = n \;\; \forall i$
- $\bigcap_{i=1}^{n} Y_i = \{(\underline{1}, [\underline{1}])\}$

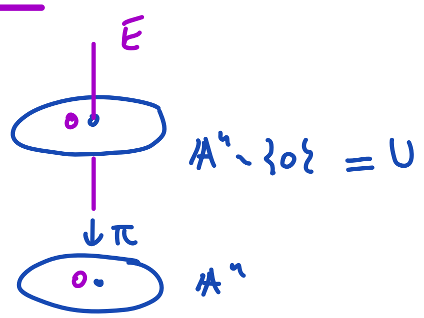
$\Big\} \underset{\text{Exercise}}{\Longrightarrow}$ $Y$ is an irreducible variety of dim $n$.

By Corollary 2: $\dim \tilde{\mathbb{A}}^n = n$

**Conclusion** $\underset{\dim n}{\tilde{\mathbb{A}}^n} \subseteq \underset{\substack{\dim n \\ \text{irred}}}{Y} \Rightarrow \tilde{\mathbb{A}}^n = Y$. is irreducible $\square$

**Exceptional locus**: $E = \pi^{-1}(0) = \{0\} \times \mathbb{P}^{r-1} \simeq \mathbb{P}^{r-1}$

**Wrong Picture**:




This picture looks reducible, but $\tilde{\mathbb{A}}^n$ is irreducible

For a better drawing, we need to see how $E$ & $U$ sit inside $\tilde{\mathbb{A}}^n$. For this, we use blow-ups of subvarieties of $\mathbb{A}^n$ meeting $(0,\cdots,0)$.

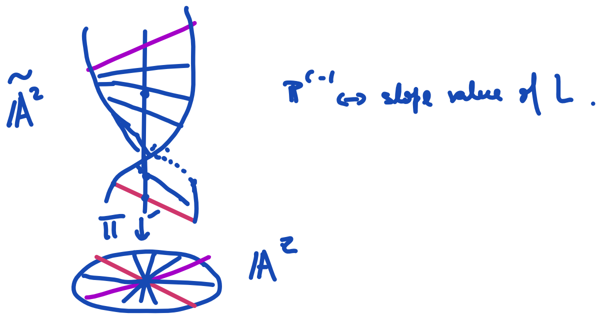
- Pick $Y = L \subseteq \mathbb{A}^n$ line through $(0,\cdots 0)$. $L = \mathbb{K}\langle v\rangle$ $v \neq 0$

Then $\tilde{Y} = \overline{\pi^{-1}(Y \setminus V(x_1, \dots x_n))} = \overline{\pi^{-1}(Y \setminus \{(0,\dots 0)\})} = \overline{\{(y:[y]) \quad y \in Y\setminus\{0\}\}}$
$\simeq L$

$\tilde{Y} \cap E = \{(0,[v])\}$

- Conclusion: $E$ parameterizes the directions in $\mathbb{A}^n$ at $0$.
- If $L$ & $L'$ are 2 different lines through $\underline{0}$, then $\tilde{L} \cap \tilde{L}' = \emptyset$.

Correct Picture:




$\tilde{\mathbb{A}}^2 =$ helix winding around the central line $\pi^{-1}(0) \simeq \mathbb{P}^1$ exactly once, so the top & bottom of the helix are identified.