Lecture XLI: Blow-ups of affine varieties III

Recall: X affine variety over $\overline{\mathbb{K}}=\mathbb{K}$ & $f_1,\ldots,f_r \in \mathcal{O}(X)$

- $\tilde{X}$ = Blow-up of X at $f_1,\ldots,f_r$ $\subseteq X \times \mathbb{P}^{r-1}$
- $E = \pi^{-1}(V(f_1,\ldots,f_r)) \subseteq \tilde{X}$ is closed [Name: exceptional locus]

Proposition: If $Y \subseteq X$ subvariety, then $\tilde{Y}$ = Blow-up of Y at $f_1|_Y,\ldots,f_r|_Y \in \mathcal{O}(Y)$ satisfies

① $\tilde{Y} \subseteq \tilde{X}$ is closed

② $\tilde{Y} = \overline{\pi^{-1}(Y \setminus V(f_1,\ldots,f_r))}$ [Name: strict transform of Y along π]

Lemma: ① If $X \not\subseteq V(f_1,\ldots,f_r)$ & X is irreducible, then $\tilde{X}$ is irreducible

② If $X = X_1 \cup \cdots \cup X_s$ irreducible decomposition, then $\tilde{X} = \tilde{X}_1 \cup \cdots \cup \tilde{X}_s$ is irreducible decomp, after removing $\tilde{X}_i = \emptyset$ corresponding to $X_i \subseteq V(f_1,\ldots,f_r)$.

Proposition: $\tilde{X} \subseteq V(I(X)\mathbb{K}[x,y] + \langle y_i x_j - y_j x_i : i,j \rangle) \subseteq X \times \mathbb{P}^{r-1}$

⚠
- In general inclusion is strict.
- Example with $=$: $X = \mathbb{A}^n$ & $\{f_1,\ldots,f_r\} = \{x_1,\ldots,x_n\}$

§1 Properties of blow-ups:

Theorem 1 (Dimension of the exceptional set) Fix $X \subseteq \mathbb{A}^n$ affine irreducible $/ \overline{\mathbb{K}}=\mathbb{K}$ & $f_1,\ldots,f_r \in \mathcal{O}(X)$ with $X \subseteq V(f_1,\ldots,f_r)$. Let $\pi: \tilde{X} \to X$ be the blow-up of X at $f_1,\ldots,f_r$. Then, every irreducible component of the exceptional set $E = \pi^{-1}(V_X(f_1,\ldots,f_r))$ has codimension 1 in $\tilde{X}$. Name: Exceptional hypersurface of the blow-up.

Proof: Enough to show every component of $E \cap U_i$ has codimension 1, where $U_i = \{(x,y) \in \tilde{X} : y_i \neq 0\}$, since $\tilde{X} = \bigcup_{i=0}^{r} U_i$.

Note: $\overline{U_i} = \tilde{X}$ if $U_i \neq \emptyset$ (U_i is open & $\tilde{X}$ is irreducible)

- **Claim:** $E \cap U_i = V_{U_i}(f_i)$

Pf/ If $(a,y) \in \tilde{X}$ & $f_i(a) \neq 0$, then $a \in U$ & so $(a,y) \in \Gamma_f$. But, $E \cap \Gamma_f = \emptyset$.
Thus $E \cap U_i \subseteq V(f_i)$.

By Proposition 1 §40.1, if $(a,y) \in U_i$ satisfies $f_i(x) = 0$, then $f_j(a) = 0\ \forall j$ ($y_i f_j(a) = y_j f_i(a)$), so $a \in V(f_1,\ldots,f_r)$ & $(a,y) \in E$.

Conclude: $V(f_i) \cap U_i = V_{U_i}(f_i) \subseteq E$.

• **Claim 2**: If $U_i \neq \emptyset$, then $f_i|_{U_i} \neq 0$ in $\mathcal{O}(U_i)$. So f_i is a non-zero divisor in U_i (irreducible)

Pf/ Otherwise, $U_i \subseteq E$ by Claim 1, so $\overline{U_i} = \tilde{X} \subseteq E$. Therefore, we get $U = \emptyset$ ie $X \subseteq V(f_1, \ldots f_r)$, which is a contradiction

By Corollary 3 § 34.1 for every irreducible component Z of E we have $Z \cap U_i \neq \emptyset$ for some i. Then, $Z \cap U_i$ is an irred component of $V_{U_i}(f_i)$ & f_i is a non-zero divisor in $\mathcal{O}(U_i)$. Thus $\operatorname{codim}_{U_i} Z = 1$.

By Lemma 4 § 31.2 $\operatorname{codim}_{\tilde{X}} Z = \operatorname{codim}_{U_i} Z \cap U_i$ $\quad$ ($U_i \subseteq \tilde{X}$ open, $Z \cap U_i \neq \emptyset$, Z irred.) $\square$

Our next goal is to prove the following theorem.

Theorem 2: The blow-up of an affine variety $X \subseteq \mathbb{A}^n$ over $\mathbb{K} = \overline{\mathbb{K}}$ at $f_1, \ldots, f_r \in \mathcal{O}(X)$ depends only on the ideal $\langle f_1, \ldots, f_r \rangle \subseteq \mathcal{O}(X)$. More precisely, if $\langle f_1', \ldots f_s' \rangle = \langle f_1, \ldots, f_r \rangle$ & $\pi : \tilde{X} \longrightarrow X$, $\pi' : \tilde{X}' \longrightarrow X$ are the corresponding blow-up maps, then there is an isomorphism $F : \tilde{X} \longrightarrow \tilde{X}'$ giving

$$\begin{tikzcd} \tilde{X} \arrow[rr, "F"] \arrow[dr, "\pi"'] \arrow[rr, phantom, shift right=4ex, "\circlearrowleft"] & & \tilde{X}' \arrow[dl, "\pi'"] \\ & X & \end{tikzcd}$$

Proof: Since $\langle f_1, \ldots, f_r \rangle = \langle f_1', \ldots, f_s' \rangle$ we can find $g_{ij} \in \mathbb{K}[x_1 \ldots x_n]$, $h_{j,k} \in \mathbb{K}[x_1 \ldots x_n]$ with $i, k \in \{1, \ldots, r\}$, $j \in \{1, \ldots s\}$ st.

$$f_i = \sum_{j=1}^{s} g_{ij} f_j' \qquad \& \qquad f_j' = \sum_{k=1}^{r} h_{jk} f_k. \qquad (*)$$

By construction: $$\underset{r \times s}{(g_{ij})} \, \underset{s \times r}{(h_{jk})} = \mathrm{Id} \in \mathrm{Mat}_{r \times r}(\mathcal{O}(X))$$

$$\underset{s \times r}{(h_{jk})} \, \underset{r \times s}{(g_{ij})} = \mathrm{Id} \in \mathrm{Mat}_{s \times s}(\mathcal{O}(X))$$

If $X \subseteq V(f_1, \ldots f_r)$, then $\tilde{X} = \tilde{X}' = \emptyset$ & there is nothing to do. Otherwise, we build $F : \tilde{X} \longrightarrow \tilde{X}'$ explicitly as the restriction of the natural map:

$$X \times \mathbb{P}^{r-1} \xrightarrow{\Phi} X \times \mathbb{P}^{s-1} \qquad (x, y) \longmapsto \left(x, \left[\sum_{k=1}^{r} h_{1,k}(\underline{x}) y_k : \cdots : \sum h_{s,k}(x) y_k\right]\right)$$

① **Bad locus**: $x \in X$, $y \in \mathbb{P}^{r-1}$ with $\sum_k h_{1,k}(\underline{x}) y_k = \cdots = \sum_k h_{s,k}(\underline{x}) y_k = 0$

On $U = X \setminus V(f_1, \dots f_r)$ we have $\tilde{X} \cap (U \times \mathbb{P}^{r-1}) = \{(x, [f_1(x) \dots f_r(x)]), x \in U\}$

Thus $[(y_i)_i] = [(f_i(x))_i]$ also on $\tilde{X}$ ie $y_i f_k(x) = y_k f_i(x)$ for $(x,y) \in X$.

- If $\sum_k' h_{j,k}(\underline{x}) y_k = 0$, then multiplying this by $f_i(x)$ we get

$$0 = \sum_k' h_{j,k}(x) y_k f_i(x) = y_i \cdot \sum_k h_{j,k}(x) f_k(x) = y_i f'_j(x)$$

Since $y \in \mathbb{P}^{r-1}$ we know $y_i \neq 0$ for some i so $f'_j(x) = 0$ on X ie $f'_j = 0 \in \mathcal{O}(X)$.

- Thus, ϕ not defined at $(x,y) \in \tilde{X}$ forces $f'_1 = \dots = f'_s = 0 \in \mathcal{O}(X)$.

The relations $(*)$ then give $f_1 = \dots = f_r = 0 \in \mathcal{O}(X)$, so $U = \phi$. & $\tilde{X} = \tilde{X}' = \phi$.

So $F = \phi|_{\tilde{X}}$ is well-defined.

(2) Image of $F \subseteq \tilde{X}'$:

Pick $(x,y) \in \pi^{-1}(U)$, so $[y_i] = [(f_i(x))_i]$

$$F(x,y) = (x, [\sum_{k=1}^{r} h_{1,k}(\underline{x}) y_k : \dots : \sum_k h_{s,k}(x) y_k]) =$$

$$= (x, [\sum_k^r h_{1,k}(x) f_k(x) : \dots : \sum_k^r h_{s,k}(x) f_k(x)])$$

$$= (x, [f'_1(x) : \dots : f'_s(x)]) \in \pi'^{-1}(U)$$

So $F(\tilde{X}) = F(\overline{\pi^{-1}(U)}) \subseteq \overline{F(\pi^{-1}(U))} \subseteq \overline{\tilde{X}'} = \tilde{X}'$

(3) F is an isomorphism: $\Psi : X \times \mathbb{P}^{r-1} \longrightarrow X \times \mathbb{P}^{r-1}$

$(x,y) \longmapsto (x, [((g_{ik}) \cdot y)_i])$

is well-defined on $\tilde{X}'$ & $\operatorname{Im} \Psi \subseteq \tilde{X}$.

Then $F^{-1} = \Psi|_{\tilde{X}}$.

(4) $\pi' \circ F = \pi$: Obvious by construction.

Remark: • By blow-up of X along $J \subseteq \mathcal{O}(X)$ ideal, we refer to the blow-up of X along any fix set of generators $\{f_1, \dots, f_r\}$ of J.

• Special case: $J = I(Y)$ for $Y \subseteq X$ subvariety. We call it the blow-up of X at Y. (Eg: $X = \mathbb{A}^n$, $J = \langle x_1, \dots x_n \rangle$, then $Y = \{0\}$.) We write it as $\mathrm{Bl}_Y X$.

Another use of blow-ups is to extend morphisms.

Proposition 1: Fix $X \subseteq \mathbb{A}^n$ affine & $f_1, \dots f_r \in \mathcal{O}(X)$. The morphism $f\ X \dashrightarrow \mathbb{P}^{r-1}$ is defined outside $V_X(f_1, \dots f_r)$. However, we can extend f to a regular morphism $\tilde{f}\ \tilde{X} \longrightarrow \mathbb{P}^{r-1}$ st

$$\begin{tikzcd} \tilde{X} \arrow[r, "\tilde{f}"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{P}^{r-1} \\ X \arrow[ur, dashed, "f"'] & \end{tikzcd}$$

Proof: $\tilde{X} \subseteq X \times \mathbb{P}^{r-1}$ so we define $\tilde{f} : \tilde{X} \longrightarrow \mathbb{P}^{r-1}$ as $\tilde{f} = p_2|_{\tilde{X}}$.

By construction: $\tilde{f}|_{\pi^{-1}(U)} = p_2|_{\Gamma_f} = f|_U$.

§2 Blow-ups of abstract varieties:

Working with open affine covers, we can extend the construction of blow-ups to arbitrary algebraic varieties. More precisely,

Fix X an algebraic variety over $\overline{\mathbb{K}} = \mathbb{K}$ & $Y \subseteq X$ a subvariety.

Set $X = U_1 \cup \dots \cup U_r$ affine open cover & define $\tilde{U}_i = \mathrm{Bl}_{Y \cap U_i} U_i$

Proposition 2: The blow-ups $\tilde{U}_i$ can be glued together to a variety $\tilde{X}$ Furthermore $X \setminus Y \hookrightarrow \tilde{X}$ open.

Proof: • We glue U_i's along $U_i \cap U_j$ to get X.

• $V_i = U_i \setminus Y \subseteq U_i$ is open & $V_i \overset{i}{\hookrightarrow} \tilde{U}_i$ is open dense.

• We glue V_i's along $V_i \cap V_j$ to get $X \setminus Y$.

• This translates to a gluing of $\tilde{U}_i$'s along $V_i \cap V_j \hookrightarrow \tilde{U}_i$ opens, to get a new space $\tilde{X}$.

• The map $X \setminus Y \hookrightarrow \tilde{X}$ is obtained by gluing $V_i \hookrightarrow \tilde{U}_i$ along $V_i \cap V_j \hookrightarrow \tilde{U}_i$ & $V_i \cap V_j \hookrightarrow \tilde{U}_j$.

Definition: We call $\tilde{X}$ the <u>blow-up of X at Y</u>.

⚠ This construction only works for subvarieties. For blow-ups at ideals, we need a modified version to be able to glue, leading to the notion of sheaf of ideals.

. For projective subvarieties we can do this directly, working with $h_1, \dots, h_r \in S(X)$ homogeneous of the same degree. Assume $X \subseteq \mathbb{P}^n$.

Set $U = X \setminus X \setminus V(f_1, \dots, f_r) \subseteq \mathbb{P}^n$

Then $\Gamma_f = \{(x, [f_1(x) : \dots : f_r(x)]) \in U \times \mathbb{P}^{r-1}\} \subseteq U \times \mathbb{P}^{r-1} \subseteq \mathbb{P}^n \times \mathbb{P}^{r-1}$.

- Since $\mathbb{P}^{r-1}$ is a variety, Γ_f is closed in $U \times \mathbb{P}^{r-1}$

. Using the Segre embedding, we see $X \times \mathbb{P}^{r-1} \subseteq \mathbb{P}^{n(r-1)+n+r-1}$ is a projective variety.

Definition: $\tilde{X} = \overline{\Gamma_f} = \overline{\{(x, [f_1(x) : \dots : f_r(x)]) \in U \times \mathbb{P}^{r-1}\}} \subseteq X \times \mathbb{P}^{r-1}$

By construction, $\tilde{X}$ is a projective variety.

§3 Tangent cones:

In $\tilde{\mathbb{A}}^n$ example, we saw that E parameterizes tangent lines of $\mathbb{A}^n$ through $\underline{o}$. This has a natural analog for $\tilde{X}$ and any $Y = \{a\}$ through tangent cones

- Fix X an algebraic variety & $p \in X$ a point. Set $\tilde{X} = Bl_p X$ & $\pi : \tilde{X} \longrightarrow X$

Then $E := \pi^{-1}(p)$ can be built from $Bl_p W$ where $W \subseteq X$ is an open affine containing p. If $W \subseteq \mathbb{K}[x_1, \dots x_n]$, then $E \subseteq \{p\} \times \mathbb{P}^{n-1} \simeq \mathbb{P}^{n-1}$

Definition: The affine cone over $E = \pi_{|_W}^{-1}(p) \subseteq \mathbb{P}^{n-1}$ is called the <u>tangent cone</u> of X at p. Notation: $C_p X$

Remark: $C_p X \subseteq \mathbb{A}^n$ is an affine variety with defining ideal $I(E) \mathbb{K}[x_1, \dots x_n]$

Lemma 1: $C_p X$ is well-defined: it is independent of the choice of affine open of X containing p up to isomorphism.

Proof: Use Theorem 2 & the fact that $E \subseteq \pi_{|W}^{-1}(p) \subseteq \tilde{W}$ for any affine open $W \subseteq X$ with $p \in W$.

Special case: If $X \subseteq \mathbb{A}^n$ is affine & $p = \underline{0}$, we view $C_p(X) \subseteq C(\mathbb{P}^{n-1}) = \mathbb{A}^n$

- As a corollary to Theorem 1 we get:

Corollary 1: $\dim C_p X = \operatorname{codim}_X \{p\}$

Proof: Replacing X with an open affine W containing p, we have
(1) $C_p X \cong C_p W$ (by Lemma 1)
(2) $\operatorname{codim}_X \{p\} = \operatorname{codim}_W \{p\}$ (by Lemma 4 §31.2)

Furthermore, we can assume every irreducible component of W meets p (otherwise, take $W' = \overline{W \setminus \bigcup_{p \notin W_i} W_i} \subseteq W$. Every component of W' is a component of W containing p since)

Case 1: If $W = \{p\} \subseteq \mathbb{A}^n$, then $C_p W = \mathbb{A}^n$, $\operatorname{codim}_W p = n$.
Case 2: If $W \subseteq \mathbb{A}^n$ & $W \neq \{p\}$, then $\dim W \geq 1$, Write $W = W_1 \cup \cdots \cup W_r$ irred decomposition. Then $\widetilde{W} = Bl_p W = \widetilde{W}_1 \cup \cdots \cup \widetilde{W}_r$ & $\widetilde{W}_i = Bl_p W_i \neq \emptyset$.
Since $\dim W \geq 1$ $\exists i$ with $\dim W_i \geq 1$. Fix such a component.
Note: $W_i \not\subseteq V(x_1 - p_1, \ldots x_n - p_n)$ since $W_i \neq \{p_i\}$ & $\overline{\mathbb{K}} = \mathbb{K}$.

By Theorem 1, every irred component Z of $E = \pi_i^{-1}(\{p\}) \subseteq \widetilde{W}_i$ has codimension one in $\widetilde{W}_i \subseteq W_i \times \mathbb{P}^{n-1}$. Thus,

Claim: $\dim Z = \dim \widetilde{W}_i - 1 = \dim W_i - 1$ (W_i & $\widetilde{W}_i$ are birational)
Pf/ $\mathbb{A}^n \times \mathbb{A}^{n-1} \simeq \mathbb{A}^{2n-1}$ is catenary (every max'l chain of irreducibles has length $2n-1$), & $\dim Z = \dim Z \cap (\mathbb{A}^n \times U_i)$ if $Z \cap (\mathbb{A}^n \times U_i) \neq \emptyset$. □

Since $C_p \widetilde{W}_i = C(E) = \bigcup_{Z \text{ cmp}} C_p Z$ is an irreducible decomposition of $C_p W_i$ then, every component of $C_p \widetilde{W}_i$ has dimension $\dim W_i$. ($\dim C(Z) = \dim Z + 1$)

Conclude: $\dim C_p \widetilde{W}_i = \max_Z \dim C_p Z = \max_Z \dim W_i = \dim W_i = \operatorname{codim}_{W_i} \{p\}$

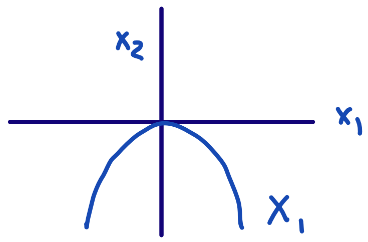
- If $\dim W_i = 0$, then $W_i = \{p\}$ & $\widetilde{W}_i = \emptyset$, so $E = \pi_i^{-1}(\{p\}) = \emptyset$ does not contribute to the tangent cone construction

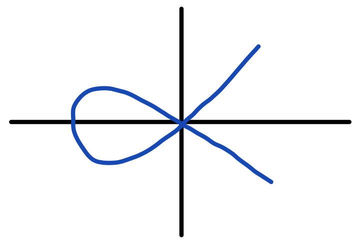
• It follows that $C_p W = \bigcup_{\dim W_i \geq 1} C_p W_i$ has dimension:

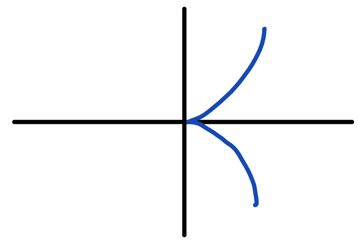
$$\dim C_p W = \max_i \{\operatorname{codim}_{W_i}\{p\} : W_i \neq \{pt\}\} \underset{\substack{\downarrow \\ \operatorname{codim}_p p = 0}}{=} \max_i \{\operatorname{codim}_{W_i}\{p\}\} = \operatorname{codim}_W \{p\}.$$

Examples: Set $\pi : \tilde{\mathbb{A}}^2 \longrightarrow \mathbb{A}^2$ & consider 3 plane curves through $p = (0,0)$

$X_1 = V(x_2 + x_1^2) \subseteq \mathbb{A}^2$ $\qquad$ $X_2 = V(x_2^2 - x_1^2 - x_1^3)$ $\qquad$ $X_3 = V(x_2^2 - x_1^3)$



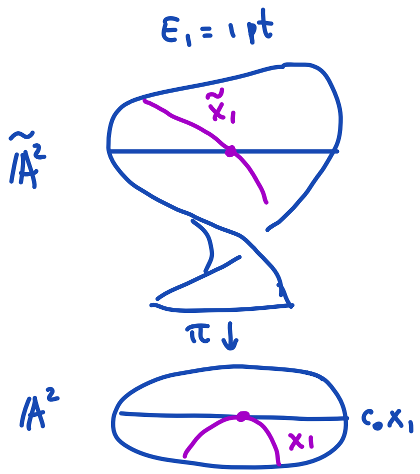


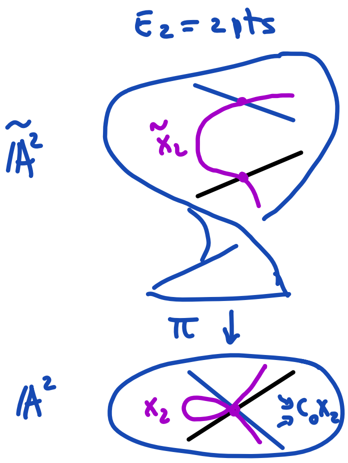


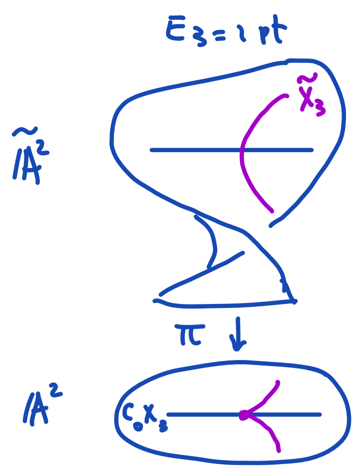
$\tilde{X}_1 = \overline{\pi^{-1}(X_1 \setminus \{\underline{0}\})}$ $\qquad$ $\tilde{X}_2 = \overline{\pi^{-1}(X_2 \setminus \{\underline{0}\})}$ $\qquad$ $\tilde{X}_3 = \overline{\pi^{-1}(X_3 \setminus \{\underline{0}\})}$

We get $\tilde{X}_1, \tilde{X}_2, \tilde{X}_3 \subseteq \tilde{\mathbb{A}}^2$ by lifting $X_i \setminus \{\underline{0}\}$ along π & adding the missing pts

The added points correspond to E_i = exceptional divisor of the blow-up of X_i at $(\underline{0})$.







Conclude: $C_0 X_1$ is 1 line, corresponding to the tangent line to X_1 at $\underline{0}$

$C_0 X_2$ is the union of 2 lines ———— lines to the 2 branches of X_2 at $\underline{0}$

$C_0 X_3$ is 1 line,

Sanity check: $\operatorname{codim}_p X_i = 1$ & $\dim C_p X_i = 1$.

Q: How do we compute $C_0 X_i$?

Example: $\tilde{\mathbb{A}}^2 = V(\langle x_1 y_2 - x_2 y_1 \rangle)$ by Example 2 § 40.2.

$\tilde{X}_2 = \{((x_1, x_2), [y_1 : y_2]) \mid (x_1, x_2) \in X \setminus \{0\}$ & $x_1 y_2 - x_2 y_1 = 0\} \subseteq \mathbb{A}^2 \times \mathbb{P}^1$

Thus $x_2^2 - x_1^2 - x_1^3 = 0$ & $x_1 y_2 - x_2 y_1 = 0$.

In particular if $(x_1, x_2) \in X_2 \setminus \{0\}$, then $\{(x_1,x_2),(y_1,y_2)\}$ are l.i.

We can write $y_1 = \lambda x_1$ & $y_2 = \lambda x_2$ for $\lambda \in \mathbb{K}^*$.

Then $\lambda^2(x_2^2 - x_1^2 - x_1^3) = y_2^2 - y_1^2 - y_1^2 x_1 = 0$ in Γ_f.

- Thus $\tilde{X}_2 = V(y_2^2 - y_1^2 - y_1^2 x_1,\ x_1 y_2 - x_2 y_1) \subseteq \mathbb{A}^2 \times \mathbb{P}^1$.
- On $E = \pi^{-1}(0)$ we have $x_1 = x_2 = 0$, so $\tilde{X}_2 \cap E = V(y_2^2 - y_1^2) \subseteq \mathbb{P}^1$

But $y_2^2 - y_1^2 = (y_2 - y_1)(y_2 + y_1)$, so $\tilde{X}_2 \cap E = \{[1:1],[1:-1]\} \subseteq \mathbb{P}^1$

Thus: $C_0 X_2 = V((x_2 - x_1)(x_2 + x_1)) \subseteq \mathbb{A}^2$ is the union of the 2 diagonal lines in the picture.

Recipe: Start with the equation for X_2 in $\mathbb{A}^2$ & toss out the higher order terms.

$\Rightarrow C_0 X_1 = V(x_2) \subseteq \mathbb{A}^2$ & $C_0 X_3 = V(x_2^2) = V(x_2)$.

Theorem 3 (Computation of tangent cones):

Fix $J \subseteq \mathbb{K}[x_1, \ldots, x_n]$ ideal & let $X = V(J) \subseteq \mathbb{A}^n$ be the associated variety. Assume $0 \in X$ & consider the blow-up $\tilde{X} \subseteq \tilde{\mathbb{A}}^n \subseteq \mathbb{A}^n \times \mathbb{P}^{n-1}$ at $x_1, \ldots, x_n$. Write the homogeneous coordinates of $\mathbb{P}^{n-1}$ as $y_1, \ldots, y_n$. Then:

(1) $\tilde{\mathbb{A}}^n = \bigcup_i V_i$ where $V_i = \tilde{\mathbb{A}}^n \cap (\mathbb{A}^n \times U_{i-1})$ $U_{i-1} = \{y_i \neq 0\} \subseteq \mathbb{P}^n$

$$V_i = \left\{ (x_i y_1 : \cdots : x_i : \cdots : x_i y_n), (y_1 : y_2 : \cdots : \underset{i\text{th}}{1} : \cdots : y_n),\ \begin{matrix} x_i \in \mathbb{K} \\ (y_1 : \cdots : 1 : \cdots y_n) \in U_i \end{matrix} \right\}$$
$\cong \mathbb{A}^n$

(2) $$Bl_0(X) \cap V_1 = V\left(\left\langle \frac{f(x_1, x_1 y_2, \ldots x_1 y_n)}{x_1^{\min\deg f}} : \begin{matrix} f \in J \\ f \neq 0 \end{matrix} \right\rangle\right) \subseteq \mathbb{A}^n$$

where $\min\deg f$ = smallest degree of a monomial in f.

(3) $E = \{0\} \times V(f^{in} : f \in J) \subseteq \{0\} \times \mathbb{P}^{n-1}$

where f^{in} = initial term in f = sum of all monomials of lowest degree in f.

Thus $C_0 X = V(f^{in} : f \in J) \subseteq \mathbb{A}^n$

⚠ $C_0 X \neq V(\langle f_j^{in} : j = 1, \ldots m \rangle)$ if $J = \langle f_1, \ldots, f_m \rangle$