MATH 8140 - T(r)opics in Algebraic Geometry – Homework 1
The Tropical semiring, valuations, polyhedra, tropical hypersurfaces

Due at 9:10am on Monday January 30th, 2017

Please indicate any source in the literature used in finding the solution to a given problem. You are encouraged to work in teams, but individual solutions **must** be submitted for grading and credit. If you work in teams, please indicate the name of your collaborator(s) for each problem.

Problem 1. The Tropical Fundamental Theorem of Algebra

(i) Prove the tropical fundamental theorem of algebra: every non-constant tropical polynomial

$$F(x) = x^{\dot{d}} \oplus c_{d-1} \odot x^{\odot(d-1)} \oplus \ldots \oplus c_0 \in \overline{\mathbb{R}}_{\text{trop}}[x]$$

when viewed as a function $F\colon \mathbb{R} \to \mathbb{R}$ factorizes as a product of linear forms $\displaystyle\bigodot_{k=1}^{d} (x \oplus \lambda_k)$.

(ii) Define the *multiplicity* of a root λ of a non-constant tropical polynomial to be the number of times it appears in the factorization from (i), that is $\text{mult}(\lambda, F) = \#\{k \in \{1, \ldots, d\} : \lambda_k = \lambda\}$.

Prove that for any $f \in \mathbb{C}\{\{t\}\}[x]$, the multiplicity of a root ω in $\text{trop}(f)$ equals the number of roots of $\text{in}_\omega(f)$ in $\mathbb{C}^*$ (counted with multiplicity).

(iii) How many roots in $\mathbb{C}\{\{t\}\}$ does the polynomial $t^3x^5 - x^2 + t^4$ have? Describe the first few terms of each series. (*Hint:* Use the Newton-Puiseux method described in the proof that $\mathbb{C}\{\{t\}\}$ is algebraically closed. You can find the proof in the notes from Lecture II.)

Problem 2. Valued field extensions through tropicalization Consider a valued field (K, val) and a finite field extension L of K. It is known that the valuation val can be extended (in at most $[L : K]^{\text{sep}}$ ways) to a valuation on L.

(i) Assuming $L = K(\alpha)$ for some α, prove that for any extension of val to L, we have that $-\text{val}(\alpha)$ is a zero of the tropical polynomial $\text{trop}(\min(\alpha, K))$, where $\min(\alpha, K)$ is the minimal polynomial of α over K.

(ii) [Exercise 2.7.3 in [MS]]† The quotient ring $L = \mathbb{Q}[s]/\langle 3s^3 + s^2 + 36s + 162\rangle$ is a field. Describe all valuations on this field that extend the 3-adic valuation on $\mathbb{Q}$.

Problem 3. Tropical plane quadrics

(i) How many combinatorial types of plane quadrics (degree two) are there? For each case, find an example of $f \in \mathbb{C}\{\{t\}\}[x, y]$ or $\overline{\mathbb{Q}_p}[x, y]$, draw the corresponding Newton subdivision and the tropical curves $\mathcal{T}f$.

(ii) Given five general points in $\mathbb{R}^2$, there exists a unique tropical quadric passing through these points. Compute and draw the tropical quadric through $(0, 5)$, $(1, 0)$, $(4, 2)$, $(7, 3)$ and $(9, 4)$.

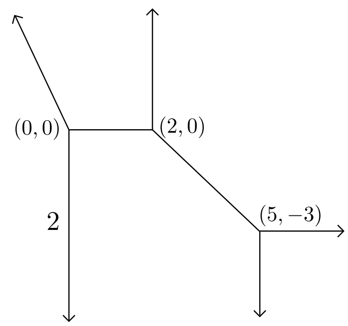
Problem 4. Prove **tropical Bézout's theorem** for transverse intersetions: two plane tropical curves C and D of degrees c and d (so they are dual to subdivisions of the 2-simplex dilated by c and d, respectively) that meet transversely have exactly $c \cdot d$ intersection points (counted with multiplicity.) The multiplicity at a point p is given by the formula:

$$\text{mult}(e)\ \text{mult}(f) \left| \det \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} \right|,$$

where e and f are the edges of C and D containing p, and $e' = (u_1, v_1)$, $f' = (u_2, v_2)$ are the primitive vectors in $\mathbb{Z}^2$ in the directions of e and f, respectively. (*Hint:* Consider $C \cup D$ as a tropical plane curve.)

† *Optional (for extra credit)*

Problem 5. Find a polynomial $f \in \mathbb{C}\{\{t\}\}[x, y]$ giving rise to the tropical plane curve below. Unless otherwise indicated, the multiplicity of an edge is assumed to be 1. The upper left ray has direction $(-1, 2)$.



Problem 6. [Exercise 2.7.10 in [MS]] Pick 2 triangles P and Q that lie in non-parallel planes in $\mathbb{R}^3$.

(i) Draw their Minkowski sum $P + Q$ and its normal fan.

(ii) Write down the f-vector of $P + Q$ (i.e., describe how many faces of each dimension does $P + Q$ have).

(iii) Verify that the normal fan of $P + Q$ is the *common refinement* of the normal fans of P and Q.

Problem 7. Draw the *tropical hypersurface* associated to each of the following Laurent polynomials over the field $\mathbb{C}\{\{t\}\}$.

(i) $f_1(x, y) = t^3\, y^3 + y^2 - xy^2 - y - t^{-1}\, xy + x^2y + t^2 + x + x^2 + t^2\, x^3$;

(ii) $f_2(x, y) = xy + 5\, xy^2 - xy^3 + t\, x^2y + 3t^2\, x^2y^2 - 7t^2x^3y$;

(iii) $f_3(x, y) = t + xy + x^{-1}y + xy^{-1} + x^{-1}y^{-1}$;

(iv) $f_4(x, y, z) = 1 + 2x + 3y + 4z$;

(v) $f_5(x, y, z) = t\, x + y + z$.

Repeat the calculation for $f_4(x, y, z)$ over $\overline{\mathbb{Q}}_2$ and $\overline{\mathbb{Q}}_3$.
(*Useful Hint:* You might want to investigate how to do some of the previous examples using the software `Gfan` or the package `tropical.lib` in `Singular`.)