MATH 8140 - T(r)opics in Algebraic Geometry – Homework 2
Gröbner basis over valued fields, matroids, tropical linear spaces

Due at 9:10am on Friday March 10th, 2017.

Please indicate any source in the literature used in finding the solution to a given problem. You are encouraged to work in teams, but individual solutions **must** be submitted for grading and credit. If you work in teams, please indicate the name of your collaborator(s) for each problem.

Problem 1. Gröbner basis computation. Given a valued field $\mathbf{K}$ with a splitting of val $a \mapsto t^a$, a *homogeneous* polynomial $f = \sum_u c_u x^u \in K[x_1, \dots, x_n]$ and $\omega \in \mathbb{R}^n$, recall that $\mathrm{in}_\omega(f) = \sum_u \overline{c_u t^{\mathrm{trop}(f)(\omega) - \langle u, \omega \rangle}} x^u \in \Bbbk[x_1, \dots, x_n]$. The following outlines how to compute an initial ideal $\mathrm{in}_\omega(I)$ when the valuation is trivial and I is homogeneous.

(i) Given any monomial ordering $\prec$, show that $\mathrm{in}_\prec \mathrm{in}_\omega I = \mathrm{in}_{\prec_\omega} I$ where $\prec_\omega$ is the monomial order refining the ω-order by $\prec$, i.e. $x^\alpha \prec_\omega x^\beta$ if $\langle \alpha, \omega \rangle < \langle \beta, \omega \rangle$ or $\langle \alpha, \omega \rangle = \langle \beta, \omega \rangle$ and $x^\alpha \prec x^\beta$.

(ii) Show that if $\{g_1, \dots, g_s\}$ is a Gröbner basis for I with respect to $\prec_\omega$, then $\mathrm{in}_\omega(g_i)$ is a Gröbner basis for $\mathrm{in}_\omega I$ with respect to $\prec$. (*Hint:* A Gröbner basis for a monomial order generates the ideal.) Conclude that $\mathrm{in}_\omega I = \langle \mathrm{in}_\omega g_1, \dots, \mathrm{in}_\omega g_s \rangle$.

(iii) [Optional] What happens if the valuation on K is non-trivial?

Problem 2. Consider the ideal $I = \langle f, g \rangle \subset \mathbb{C}\{\{t\}\}[x^{\pm}, y^{\pm}]$ where

$$f = t^2 x^2 + xy + t^2 y^2 + x + y + t^2 \quad \text{and} \quad g = 5 + 6t\, x + 17t\, y - 4t^3\, xy.$$

(i) For each $\omega \in \mathrm{Trop}(V(f)) \cap \mathrm{Trop}(V(g))$, compute $\mathrm{in}_\omega I$. Is $\{f, g\}$ a tropical basis for I?

(ii) There are four points in the variety $V(I) \subset (\mathbb{C}\{\{t\}\}^*)^2$. Compute the leading term of each point.

Problem 3. Consider the linear ideal $I = \langle x_1 + x_2 + x_3 + x_4 + x_5, 3x_2 + 5x_3 + 7x_4 + 11x_5 \rangle \subset \mathbb{C}[x_1^{\pm}, \dots, x_5^{\pm}]$. The tropical variety $\mathrm{Trop}(I)$ is a three-dimensional fan with a one-dimensional lineality space. It is a fan over the complete graph K_5. The fan has ten maximal cones and five codimensional-one cones. The following shows that a change of coordinates in $T = (\mathbf{K}^*)^5$ might change the structure of the tropical variety.

Consider the automorphism $\varphi^* \colon T \to T$ defined by $x_1 \mapsto x_1$, $x_2 \mapsto x_2 x_3$, $x_3 \mapsto x_3 x_4$, $x_4 \mapsto x_4 x_5$, $x_5 \mapsto x_5$ and let $J = (\varphi^*)^{-1}(I) \subset \mathbb{C}[x_1^{\pm}, \dots, x_5^{\pm}]$.

(i) Show that $\mathrm{Trop}(J)$ has the same support as $\mathrm{Trop}(I)$.

(ii) Show that the Gröbner structure of $\mathrm{Trop}(J)$ has 12 maximal cones, obtained as the cone over a subdivision of K_5 where 2 edges are subdivided. (*Hint:* You can use `Gfan` to verify this.)

Problem 4. Tropicalization of linear spaces. Let V be the row space of the following matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix}$$

defined over $\mathbf{K} = \mathbb{C}\{\{t\}\}$. Let M be the matroid of columns of A with groundset $[6] := \{1, 2, \dots, 6\}$.

(i) Let $I(V)$ be the linear ideal in $K[x_1, \dots, x_6]$ defining $V \subset \mathbb{P}^5$. Compute $I(V)$, list the circuits of $I(V)$ and M, respectively.

(ii) Draw the Hasse diagram of the lattice of flats of M and show that the flats of M are in correspondence with partitions of the set $\{2,3,4,5\}$.

(iii) Compute the tropical variety $\operatorname{Trop} V/\mathbb{R}\cdot\mathbf{1} \subset \mathbb{R}^5/\mathbb{R}\cdot\mathbf{1}$. Prove that it is homeomorphic to a cone over the Petersen graph.

(iv) For each $i = 1,\ldots,6$ let H_i be the hyperplane in $\mathbb{P}^3_K$ with normal vector $a_i = i^{\text{th}}$. column of A. Let $X = \mathbb{P}^3_K \smallsetminus \bigcup_{i=1}^{6} H_i$ be the hyperplane complement. Show that the map

$$\varphi\colon X \to \mathbb{P}^5_K \qquad \mathbf{x} = [x_1 : \ldots : x_6] \mapsto [a_1\cdot\mathbf{x} : \ldots : a_6\cdot\mathbf{x}] \in (K^*)^6/K^* \simeq (K^*)^5$$

is injective and identifies its image with V.

(v) Check that the hyperplanes $\{H_i\}_{i=1}^6$ from item (iv), ordered by inclusion, form a partially ordered set that is dual to the lattice of flats of the matroid M. Conclude that the tropical variety records the information of "what is missing" from X.

Problem 5. Given the lattice of flats of a matroid M (not necessarily realizable), describe a method to recover the circuits of M, the independent sets of M and the bases of M. Illustrate this method with the matroid from Problem 4.

Problem 6. Characteristic dependence of $\mathbf{TropGr(3,7)}$.

Consider the weight vector $\omega = -(e_{124}+e_{235}+e_{346}+e_{457}+e_{156}+e_{267}+e_{137}) \in \mathbb{R}^{\binom{7}{3}}$ corresponding to the negative incidence vector for the lines in the *Fano plane*. Let

$$f = 2\,p_{123}p_{467}p_{567} - p_{367}p_{567}p_{124} - p_{167}p_{467}p_{235} - p_{127}p_{567}p_{346} - p_{126}p_{367}p_{457} - p_{237}p_{467}p_{156}$$
$$+\, p_{134}p_{567}p_{267} + p_{246}p_{567}p_{137} + p_{136}p_{267}p_{457}.$$

(i) Show that $f \in I_{3,7}$ (the Plücker ideal defining $\mathbb{G}\mathrm{r}(3,7)$).

(ii) Compute $\mathrm{in}_\omega(f)$ and conclude that $\omega \notin \operatorname{Trop}\mathbb{G}\mathrm{r}(3,7)$ if $\mathrm{char}(\Bbbk) \neq 2$.

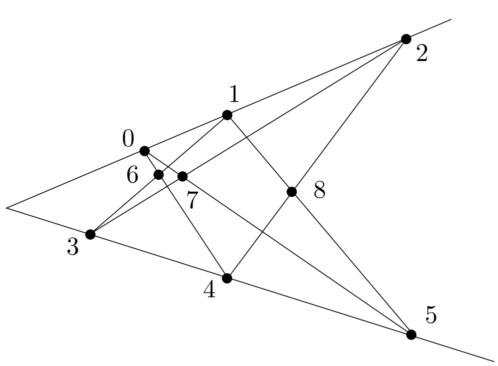
(iii) Show that if $\Bbbk = \mathbf{F}_2$ (for example for $\mathbf{K} = \mathbb{Q}_2$) then $\omega \in \operatorname{Trop}\mathbb{G}\mathrm{r}(3,7)$.

(iv) Consider the weight vector $\omega' = \omega + e_{124}$. Then show that $\mathrm{in}_{\omega'}(f)$ is a monomial if $\mathrm{char}(\Bbbk) = 2$.

(v) Show that if $\mathrm{char}(\Bbbk) = 0$, $\mathrm{in}_{\omega'}(I_{3,7})$ does not contain a monomial (*Hint:* You might want to explore how to do this using `Gfan` or the `Macaulay2` build-in command `leadTerm`*.

(vi) Show that both ω and ω' lie in Dr_M where M is the Fano matroid.

Problem 7. The non-Pappus matroid. The *non-Pappus* matroid is the rank 3 matroid on $\{0,\ldots,8\}$ with circuits 012, 046, 057, 136, 158, 237, 248, 345 plus every subset of size four not containing one of these triples.



(i) Show that this matroid is not realizable over any field, as Pappus' Theorem implies that any realizable matroid with these circuits also has the circuit 678, i.e. the points labelled $6,7,8$ are collinear.

(ii) Compute the f-vector of the matroid polytope of the non-Pappus matroid.

(iii) Describe the tropical linear space $\operatorname{Trop}(M) \subseteq \mathbb{R}^8$, i.e. the Bergman fan of the non-Pappus matroid M.

(iv) Show directly that there is no variety $X \subseteq (K^*)^8$ with $\operatorname{Trop}(X) = \operatorname{Trop}(M)$.

Problem 8. Compute the Dressian Dr_M for the non-Pappus matroid M from Problem 7.

*`http://www.math.uiuc.edu/Macaulay2/doc/Macaulay2-1.9/share/doc/Macaulay2/Macaulay2Doc/html/___Weights.html`