MATH 8140: TOPICS IN ALGEBRAIC GEOMETRY

Lecture 1 (01/09/17): Introduction / Motivation for Tropical Geometry

§1 Why?

SLOGAN 1: Tropical varieties are combinatorial shadows (polyhedral complexes) of algebraic varieties over non-Archimedean valued fields & their Berkovich analytifications.

$\rightsquigarrow$ Compute geometric invariants (dimension, degree of projective varieties) by working on the "easier" polyhedral side.

- Find nice compactifications / combinatorial degenerations
- Structures w/ interesting combinatorics, including matroids (eg Rota's conjecture)

SLOGAN 2: Tropical Geometry is algebraic geometry over the tropical semiring $(\overline{\mathbb{R}}, \oplus, \odot)$

$\rightsquigarrow$ Tropical schemes (current thread ...)

Two aspects: Course Outline

PART (I): Embedded tropicalizations $\hookrightarrow X \subseteq \mathbb{G}_m^n, \mathbb{A}^n, \mathbb{P}^n$ or other toric vars.

PART (II): Abstract tropical varieties

tropicalization map

BERKOVICH ANALYTIFICATIONS

Part I:

1. Interesting initial (combinatorial) degenerations of X $\leftrightarrow$ Gröbner basis
2. Structure Thm / Fundamental Theorem of trop geometry
3. Central example: (moduli of) tropicalized linear spaces $\leftrightarrow$ Tropical Grassmannian
4. Enumerative Geometry:
   - tropical Caporaso-Harris, WDVV eqns
   - Mikhalkin's Correspondence Theorems
5. Tropical Compactifications [Tevelev]
6. Toric degenerations [GROSS-SIEBERT PROGRAM]

[Computations w/ M2, Singular, Gfan, Sage, ...]

Part II:

1. Main example: Moduli of abstract tropical curves $\leftrightarrow$ $\overline{\mathcal{M}}_{g,n}, \mathcal{M}_g$
2. Realizable vs non-realizable (eg matroids, Grassmannian vs Dressian)
3. Tropical Intersection Theory $\updownarrow$
   - Stable Intersections
   - Tropical Bernstein's Thm.

§2 Tropical semiring:

Def $\overline{\mathbb{R}}_{trop} = (\overline{\mathbb{R}}, \oplus, \odot)$ is the max-plus semiring, where

(1) $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$, (2) $\oplus = \max$ ($x \oplus y = \max\{x, y\}$), (3) $\odot$ = usual $+$ ($x \odot y = x + y$)

Note: Max-Plus Ring $\overset{\sim}{\rightsquigarrow}$ Min Plus ring, $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$, $\oplus = \min$

$x \longmapsto -x$

Eg: $3 \oplus 5 = 5$, $3 \odot 5 = 8$

Axioms:
- Associativity
- Commutativity
- Neutral elements: $0_{trop} = -\infty$, $1_{trop} = 0$.
- Distributive Laws: $x \odot (y \oplus z) = x \odot y \oplus x \odot z$.

Note: $\oplus$ doesn't have an inverse, eg $4 \oplus x = 0_{trop}$ has no solution.

$\rightsquigarrow$ tropical (max-plus) semiring.

Extra properties: $x \oplus 0 = \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$, $x \odot (-\infty) = -\infty$.

$\rightsquigarrow$ Tropical Linear algebra / tropical convexity [Sections §5.1–5.3 of [MS]]
w/ applications to Dynamical Programming (Assignment Problem, Comp Bio, Econ)

$\rightsquigarrow$ Tropical polynomials:

- Monomials: $a \odot x_1^{\odot m_1} \odot \cdots \odot x_n^{\odot m_n} = a + m_1 \cdot x_1 + m_2 x_2 + \cdots + m_n x_n$

$$= \underset{\overline{\mathbb{R}}}{\underset{\cap}{a}} + \langle \underset{\mathbb{Z}^n_{\geq 0}}{\underset{\cap}{\underline{m}}}, \underset{(\overline{\mathbb{R}})^n}{\underset{\cap}{\underline{x}}} \rangle \longleftrightarrow \text{affine linear function w/ integer slopes.}$$

- Tropical polynomial = finite combination of tropical monomials:

$$P(x_1, \dots, x_n) = \bigoplus_{\substack{\alpha \\ \text{finite}}} c_\alpha \odot \underline{x}^\alpha = \underline{\max}\{c_\alpha + \langle \alpha, x \rangle\}$$

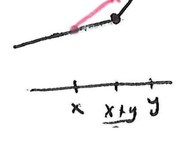
So. $p : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a function satisfying:

(1) p is continuous

(2) p is piecewise linear with a finite number of pieces

(3) p is convex: $p\left(\frac{1}{2}(x+y)\right) \leq \frac{1}{2}(p(x) + p(y))$ for all x, y in $\mathbb{R}^n$.

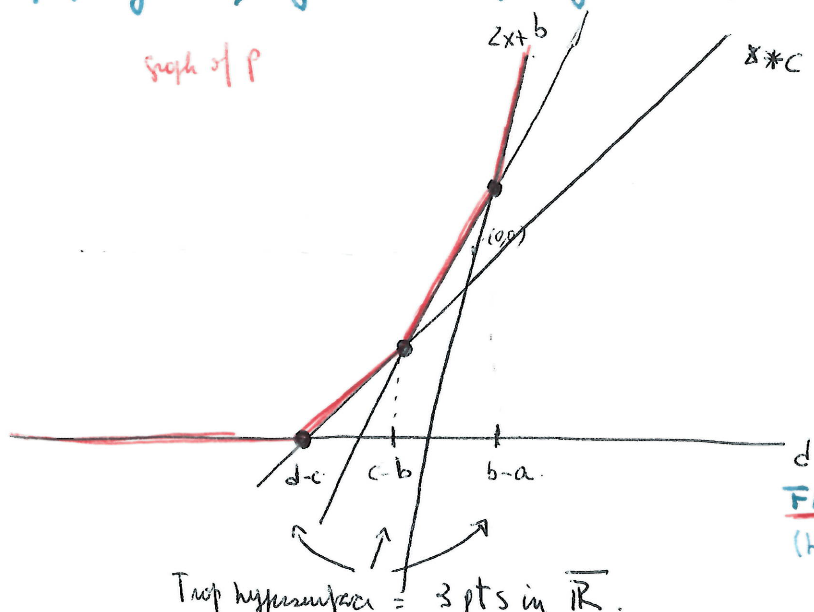
[if min-plus, replace (3) w/ p is concave]



Example: $n=1$ $\quad P(x) = a\odot x^3 \oplus b\odot x^2 \oplus c\odot x \oplus d$

$$= a\odot\left(x^3 \oplus (b-a)\odot x^2 \oplus (c-a)\odot x \oplus (d-a)\right)$$ ⇝ can assume $a=0$

graph $y=3x$, $y=2x+b$, $y=x+c$, $y=d$ & take the upper envelope.



Exercise All 4 lines contribute if

$$b-a \geq c-b \geq d-c$$

[The breakpoints where p is non-linear]

⇕

$$P(x) = a\odot(x\oplus(b-a))\odot(x\oplus(c-b))\odot(x\oplus(d-c))$$

FACT (HW1): The Fundamental Thm of Algebra holds tropically (ie The function P factors into linear functions)

Eg $x^2\oplus(-17)\odot x\oplus(-2) \underset{\text{same function}}{=} x^2\oplus(-1)\odot x\oplus(-2) = (x\oplus(-1))^2$.

Def Tropical hypersurface in $\mathbb{R}^n$ defined by $P = \{w\in\mathbb{R}^n \mid \max(\text{defining } P(w))$ is attained twice$\}$

$= \{w\in\mathbb{R}^n \mid P$ is not linear at $w\}$.

§3 Plane curves:

$n=2$: $\quad P(x,y) = \bigoplus_{(i,j)} c_{ij}\odot x^i\odot y^j$

Prop: The curve $\mathcal{T}(P)$ is a finite graph, embedded in $\mathbb{R}^2$, satisfying:

(1) contains bounded & unbounded edges with weights (trop. multiplicities) in $\mathbb{Z}_{>0}$

(2) all edge slopes are rational

(3) balancing (0-tension) condition around ~~each~~ vertex

↳ edges around vertex = primitive (in $\mathbb{Z}^2$) vector $\vec{n}_e$ generating each edge w/ multiplicity $m_e\in\mathbb{Z}_{>0}$

Balancing at v = $\sum_{v\in e \text{ edge}} m_e \vec{n}_e = \vec{0}$ in $\mathbb{Z}^2$

Proof: Structure Thm for tropical varieties.

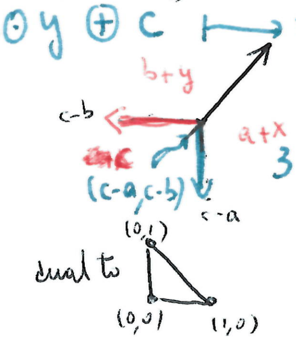
Example Tropical line (degree 1) $\quad P(x,y) = a\odot \underset{= x^1 \odot y^0}{x} \oplus b\odot \underset{= x^0\odot y^1}{y} \oplus \underset{= x^0\odot y^0}{c} \longmapsto \max\{x+a, y+b, c\}$

(for min: ⅄)

— $x+a = c \geq y+b \iff x=c-a,\ y\leq c-b$

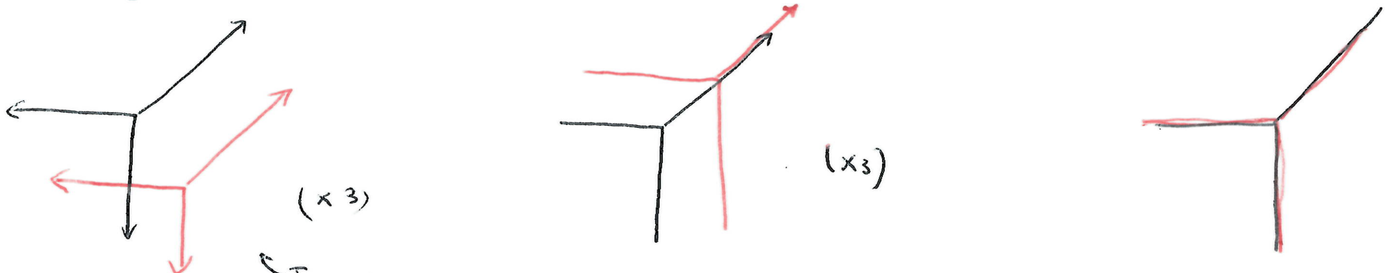
— $x+a = y+b \geq c \iff x\geq c-a \quad y = x+(a-b)\geq c-b$

— $c = y+b \geq x+a \iff y=c-b,\ x\leq c-a$



3 fixed directions $\{(1,1), (-1,0), (0,-1)\}$

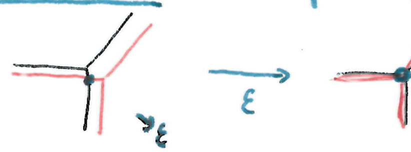
• Intersecting 2 tropical lines in $\mathbb{R}^2$: 3 possible scenarios:



1 point (same as classical) — 1 half-ray — coincide

Solution: Displacement rule ⇝ Stable Intersection in tropical geometry

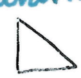
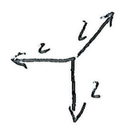
$\lim_{\underline{\varepsilon}\to(0,0)} L_1 \cap (\underline{\varepsilon} + L_2) =: L_1 \cap_{st} L_2$

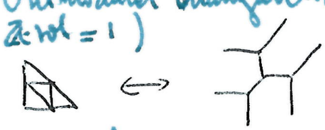
Eg  vertex is the stable intersection

• Higher degree? Other p's?

$\mathcal{T}(p)$ = dual to a regular subdivision of the "Newton polytope" = conv hull $\{(i,j) \mid (i,j) \in \mathrm{supp}(p)\}$

$p = \bigoplus_{(i,j)} c_{ij} \odot x^{\underline{i}}$ Eg deg = 2: 2 extremal cases for tropical plane quadrics  $= 2\Delta_2$

CASE 1: no subdivision.  

CASE 2 Unimodular triangulation ($\mathbb{Z}$-vol = 1) 

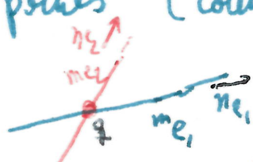
Exercise 2: Find the remaining combinatorial types of tropical plane quadrics.

Def p has degree d if its Newton polytope $= d\Delta_2$ = conv hull $\{(0,0), (d,0), (0,d)\}$

Classical Bézout: If C_1 has degree d_1, C_2 has degree d_2, then C_1 meets C_2 at $d_1 d_2$ points (counted w/ multiplicity)

The same theorem holds tropically.

Tropical Bézout: Let $\mathcal{T}p_i$ be of degree d_i ($i=1,2$). If $\mathcal{T}p_1$ & $\mathcal{T}p_2$ meet transversally then $\mathcal{T}p_1 \cap \mathcal{T}p_2$ consists of $d_1 d_2$ points (counted w/ multiplicity)

• multiplicity of an intersection pt q 

$\begin{cases} q \in \text{interior of an edge } e_1 \text{ of } \mathcal{T}p_1 \\ \quad\text{———}\quad e_2 \text{ of } \mathcal{T}p_2 \end{cases}$ $\Rightarrow$ multiplicity count of $q = m_{e_1} m_{e_2} \left|\det\begin{pmatrix}\vec{n}_{e_1} & \vec{n}_{e_2}\end{pmatrix}\right|$

$\mathcal{E}$ (transverse intersection)

• If the intersection is not transverse, replace with stable intersection.

Eg above $d_1 = d_2 = 1$ $n_{e_1} = (-1,0)$, $n_{e_2} = (0,-1)$ $\Rightarrow$ mult$(q) = 1\cdot 1 \left|\det\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\right| = 1$ ✓

$m_{e_1} = 1$ $m_{e_2} = 1$