Lecture II: Non-Archimedean valued fields

References: • "Valued fields" (Engler-Prestel)
• "Commutative Algebra vol II" (Zariski-Samuel)

§1 Basics on valuations: Fix K a field

Def: $(\Gamma, +, 0)$ abelian group is ordered if $\leq$ is an order on Γ satisfying
$\delta \leq \gamma \Rightarrow \delta+\lambda \leq \gamma+\lambda \quad \forall \lambda \in \Gamma$ (ie addition is MONOTONE)

Eg: $\Gamma = \mathbb{Z}$, $\mathbb{Z}^n_{lex}$ ($\underline{x} \leq_{lex} \underline{y}$ if first non-zero entry in $\underline{y}-\underline{x}$ is ≥ 0)
$\rightsquigarrow \Gamma = \mathbb{R}^n_{lex}$ also works.

Thm [Hahn] These are essentially all: given Γ $\exists$ set I and an order preserving map $\Gamma \overset{i}{\hookrightarrow} \mathbb{R}^{(I)}_{lex}$ (order pres $\Rightarrow$ injective)

Def: $v: K^\times \twoheadrightarrow \Gamma$ is a valuation on K if Γ is an ordered ab. gp & v satisfies:
(1) $val(ab) = val(a) + val(b) \quad \forall a,b$.
(2) $val(a+b) \geq \min\{val(a), val(b)\} \quad \forall a,b \neq 0, a+b \neq 0$

Notice $im(v) \subseteq \Gamma$ is a group (ab gp & ordered), called the valuation group. Wlog, we assume v is SURJECTIVE ($\Gamma_{val} = v(K^\times)$)

Extend $v: K \twoheadrightarrow \Gamma \cup \{\infty\}$ by $v(0) = \infty$ (& $0 \in K$ is the only element w/ valuation $= +\infty$).

NOTE: $v(1) = v(1 \cdot 1) = 2v(1) \Rightarrow v(1) = 0$.

• If $0 \neq \Gamma \hookrightarrow \mathbb{R}$ we say v has rank 1. This will always be the case in this course

For tropicalizations / analytifications for higher rank valuations, see [Foster-Ranganathan] series of papers

Def • $\mathcal{O}_v = \{x : v(x) \geq 0\}$ is its valuation ring $(= K^\circ)$
($x \notin \mathcal{O}_v \Rightarrow v(x) < 0 \overset{-val(x)}{\Rightarrow} v(x^{-1}) > 0 \Rightarrow x^{-1} \in \mathcal{O}_v$ [usual defn. of valn ring])

• $\mathfrak{m}_v = \{x : v(x) > 0\} \subset \mathcal{O}_v$ is the unique maximal ideal of $\mathcal{O}$ $(= K^{\circ\circ})$
($v(x) = 0 \Rightarrow 0 = v(1) = v(xx^{-1}) = v(x^{-1}) \Rightarrow x^{-1} \in \mathcal{O}_v$)

Note: $v(1) = 0 \Rightarrow v(\mathcal{O}^\times) = 0$

• $\Bbbk = \mathcal{O}_v / \mathfrak{m}_v$ $(= \widetilde{K})$ is a field (the residue field of (K, v))

KEY LEMMA: $v(a) < v(b) \Rightarrow v(a+b) = v(a)$ (ie min in (2) is achieved!)

Pf/ $v(b) = v(-b) \Rightarrow v(a) = v(a+b-b) \geq \min\{v(a+b), v(b)\} \geq \min\{v(a), v(b)\} = v(a)$
So $\geq$ are $=$ & $v(a+b) = v(a)$. $\square$

Def: Two valns v_1, v_2 on K are <u>equivalent</u> if $\mathcal{O}_{v_1} = \mathcal{O}_{v_2}$.

Proposition: $v_i : K \twoheadrightarrow \Gamma_i \cup \{\infty\}$ are equiv. valns $\iff \exists\, \rho : \Gamma_1 \to \Gamma_2$ order preserving iso s.t. $\rho \circ v_1 = v_2$.

Pf/ $(\Rightarrow)$ $\sigma_i : \dfrac{K^\times}{\mathcal{O}^\times} \twoheadrightarrow \Gamma_i$ $\quad \sigma_i(x\mathcal{O}^\times) := v_i(x)$ is well-defined.
$\mathcal{O} = \mathcal{O}_{v_1} = \mathcal{O}_{v_2}$

. $K^\times/\mathcal{O}^\times$ is an ab. gp under mult w/ unit 1. $\quad (x\mathcal{O}_v^\times + y\mathcal{O}_v^\times = xy\mathcal{O}_v^\times)$

$\begin{cases} \cdot \text{ order} : x\mathcal{O}^\times \leq y\mathcal{O}^\times \text{ iff } \frac{y}{x} \in \mathcal{O} \\ \text{(total order b/c } \mathcal{O} \text{ is a valuation ring)} \\ \cdot \text{ multiplication is monotone} \end{cases}$ $\quad$ (well-defined ✓) $\quad [x\mathcal{O}^\times \geq 1\mathcal{O}^\times \iff x \in \mathcal{O}]$

$\Rightarrow \rho = \sigma_2 \circ \sigma_1^{-1}$ works.

$(\Leftarrow)$ Follows because $v_1(x) \geq 0 \iff v_2(x) \geq 0$ $\quad$ [ρ preserves order & its iso] $\square$

. Unless $v : K^\times \twoheadrightarrow \Gamma \leq \mathbb{R}$ is trivial, we will <u>ALWAYS assume $1 \in \Gamma$</u> ($\exists\, v' \sim v$ with this property)

§2 Examples

⓪ Trivial valuation on any field $v(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases}$

① $v_p : \mathbb{Q}^\times \twoheadrightarrow \mathbb{Z}$ p-adic valuation (p prime) $\quad \frac{a}{b} = p^{s}\frac{a'}{b'}$ $\quad p \nmid a'b' \rightsquigarrow v_p(\frac{a}{b}) = s$

$\rightsquigarrow$ p-adic norms on $\mathbb{Q}$ $\quad |x| = p^{-v_p(x)}$ $\quad$ (only non-trivial valns on $\mathbb{Q}$)

In general $|\ | : K \longrightarrow \mathbb{R}_{\geq 0}$ $\quad \begin{cases} |x| = 0 \iff x = 0 \\ |xy| = |x||y| \quad \text{(multiplicative)} \\ |x+y| \leq \max\{|x|, |y|\} \quad \text{ULTRAMETRIC; non-Arch } \Delta\text{-inequality)} \\ \quad = \text{ if } \neq \end{cases}$

absolute value

$\updownarrow$

$\mathrm{val} : K^\times \longrightarrow \mathbb{R}$ $\quad \mathrm{val}(x) = -\log|x|$ valn. on K. $\quad |n| = |\underbrace{1 + \cdots + 1}_{n \text{ times}}| \leq |1| = 1 \ \forall n$

. Topology on K induced by $|\ |$ is totally disconnected (because every open ball is closed)
$\rightsquigarrow$ Berkovich spaces: add pts to K (or varieties in K^n) to repair the topology

. Take completion $\hat{K}$ w.r.t to this topology $|\ | : \hat{K} \longrightarrow \mathbb{R}_{\geq 0}$ $\quad (|\lim_\alpha x_\alpha| = \lim_\alpha |x_\alpha|)$
($\hat{K}$ is a field containing K) $\quad$ (x_α) Cauchy seq.

$\rightsquigarrow$ $\boxed{\text{Extend } v : \hat{K} \longrightarrow \mathbb{R}}$

For v_p: $\hat{\mathbb{Q}} := \mathbb{Q}_p$.

Eg. $p = 2$ $\quad v_2(8/9) = v_2(2^3/9) = 3$, $\quad v_2(\frac{181}{16}) = -4$.

Rmk. $\mathcal{O}_{v_p} = \{\frac{a}{b} \mid p \nmid b\} \supset m_{v_p} = \{p\frac{a}{b} \mid p \nmid b\} = (p)\mathcal{O}_{v_p}$ $\rightsquigarrow$ $\boxed{k = \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}}$

② $f \in k[x]$ irreducible $\Rightarrow$ f-adic valn on $k(x)$ $v\left(\frac{g}{h}\right) = v\left(f^s \frac{g_1}{h_1}\right) = s$ $f \nmid g_1, h_1$.

③ MAIN EXAMPLE Field of Puiseux series $\mathbb{k}\{\{t\}\}$. over $\mathbb{k}$ ($\mathbb{k}$ with trivial valuation) e.g. $\mathbb{C}$)

Def $c_{(t)} = c_1 t^{a_1} + c_2 t^{a_2} + \ldots\ldots$ where $c_i \in \mathbb{k}$ $\forall i \in \mathbb{N}$, AND $a_1 < a_2 < \ldots\ldots$ are in $\mathbb{Q}$ & with common denominator.

So $\mathbb{k}\{\{t\}\} = \bigcup_{n \geq 1} \mathbb{k}((t^{1/n}))$

Valuation: t-adic one: $v(c_{(t)}) = a_1$ (lowest exponent)

Fact: It's the order of $t=0$ as a zero or pole of $c(t)$.

Theorem 1: $\mathbb{k}\{\{t\}\}$ is algebraically closed if $\overline{\mathbb{k}} = \mathbb{k}$ & char $\mathbb{k} = 0$.

Note: False if char $\mathbb{k} = p > 0$ [Artin-Schreier poly $x^p - x - t^{-1}$ has no roots]

Proof idea: Newton-Puiseux's algorithm (see end of the notes)

④ GENERALIZED POWER SERIES ⇝ MOST GENERAL ONES!

Def $\overline{\mathbb{k}} = \mathbb{k}$ w/ trivial valn, $G \subset \mathbb{R}$ divisible group. Then $K = \mathbb{k}((G))$ Malcev-Neumann ring is $\{ \alpha_{(t)} = \sum_{g \in G} \alpha_g t^g \mid \alpha_g \in \mathbb{k}$ $\underline{\forall} g$ & $\text{supp}(\alpha) = \{g \in G : \alpha_g \neq 0\}$ is well-ordered$\}$

⇝ can add ✓ & multiply $\left(\left(\sum_g \alpha_g t^g \cdot \sum_{g'} \beta_{g'} t^{g'}\right) = \sum_h \left(\sum_{g+g'=h} \alpha_g \beta_{g'}\right) t^h\right)$

$\text{supp}(a+b) \subseteq \text{supp}(a) \cup \text{supp}(b)$

$\{(g+g') : g \in \text{supp}\, \alpha, g' \in \text{supp}\, \beta\}$ is well-ordered

$\therefore \{(g+g') : g+g' = h\}$ is finite $\forall h \in G$

• Same reasoning, can build $\alpha_{(t)}^{-1}$ if $\alpha_{(t)} \neq 0$.

• ∃ t-adic valuation on K ✓

Thm 2: $K = \overline{K}$ [see [Poonen 93]: "Maximally complete fields"]

Thm 3: Fix G divisible gp & (K, val) valued field s.t. $\overline{\mathbb{k}} = \mathbb{k}$ & $\Gamma_{\text{val}} = G$.
If val is trivial on prime field of K ($= \mathbb{F}_p$ or $\mathbb{Q}$), then $(K, \text{val}) \hookrightarrow \mathbb{k}((G))$
& $\text{val} = v_{\mathbb{k}((G))}|_K$ (t-adic valuation on K is val).

⇒ Everything viewed in some $\mathbb{k}((G))$

• Back to Artin-Schreier polynomial: $x^p - x - t^{-1}$

Roots = $\left(t^{-1/p} + t^{-1/p^2} + t^{-1/p^3} + \ldots\right) + c$ for $c \in \mathbb{F}_p \subseteq \mathbb{k}$. ⇝ Not in $\mathbb{k}\{\{t\}\}$

Why G divisible?

LEMMA: $K = \overline{K}$ w/ non-trivial valn. Then $\Gamma_{\text{val}} \subseteq \mathbb{R}$ is a divisible group & dense in $\mathbb{R}$.

Pf/ $\text{val}(a^{1/n}) = \frac{1}{n}\text{val}(a)$ ⇝ Γ divisible ✓. Since we assume $1 \in \Gamma_{\text{val}} \Rightarrow \mathbb{Q} \subseteq \Gamma_{\text{val}}$ ⇒ dense in $\mathbb{R}$ □

Proof of Thm 1: Given $F_{(x)} = \sum_{i=0}^{n} c_i x^i \in K[x]$ $\quad (K = \Bbbk\{\{t\}\})$, we construct a root of F in K term by term, in an algorithmic fashion.

Assumptions:

(1) $\operatorname{val}(c_i) \geq 0 \quad \forall i$

(2) $\exists\, j$ with $\operatorname{val}(c_j) = 0$.

(3) $c_0 \neq 0$ (o/w $F(0)=0$ ✓)

(4) $\operatorname{val}(c_0) \geq 0$

(1)–(3): easy: $\dfrac{t^{-\min\{\operatorname{val}(c_i) : 0\leq i\leq n\}}\, F_{(x)}}{x^m}$ $\quad m = \min\{i : c_i \neq 0\}$ has this property.

For (4): If $\operatorname{val}(c_n) \geq 0 \rightsquigarrow G = x^n F(\tfrac{1}{x}) = \sum_{i=0}^{n} c_{n-i} x^i$ has (1)–(4) properties.

And if $\lambda \neq 0$ is a root of $G \Rightarrow \frac{1}{\lambda}$ is a root of F.

Assume $\operatorname{val}(c_0) = \operatorname{val}(c_n) = 0$ $\quad \in \mathcal{O}_v^{\times}$

Notice $\mathcal{O}_v / \mathfrak{m}_v = \Bbbk$ & $F \in \mathcal{O}_v[x]$ so $f := \overline{F} \in \Bbbk[x]$ (mod $\mathfrak{m}_v \mathcal{O}_v[x]$) has degree n & $f(0) \neq 0$.

Since $\overline{\Bbbk} = \Bbbk$, $f(\lambda) = 0$ for some $\lambda \in \Bbbk^{\times}$ & $\operatorname{val}(\lambda) = 0$.

$F(x) = f(x) + g(x)$ with $g \in \mathfrak{m}_v \mathcal{O}_v[x]$ $\quad g = \sum_{i=0}^{r} \underset{\in \mathfrak{m}_v}{a_i} x^i$ $\quad \operatorname{val}(g_{(\lambda)}) \geq \min\{\overset{>0}{\operatorname{val}(a_i)} + i \operatorname{val}(\lambda)\} > 0$.

$F(\lambda) = \underbrace{f(\lambda)}_{=0} + g(\lambda) \rightsquigarrow \operatorname{val}(F(\lambda)) > 0$

Write $\widetilde{F}(x) = F(x+\lambda) = \sum_{i=0}^{n} \left(\sum_{j=i}^{n} c_j \binom{j}{i} \lambda^{j-i} \right) x^i \in \mathcal{O}_v[x]$.

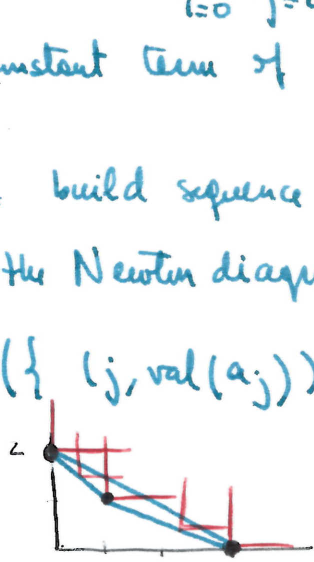
& $\widetilde{F}(0) = F(\lambda)$ = constant term of $\widetilde{F}$ has $\operatorname{val}(F(\lambda)) > 0$. $\Rightarrow$ (4) holds for $\widetilde{F}$.

Step 2: Set $F_0 = F$ & build sequence $F_\ell = \sum_{i=0}^{n} c_i^{(\ell)} x^i$ satisfying (1) — (4)

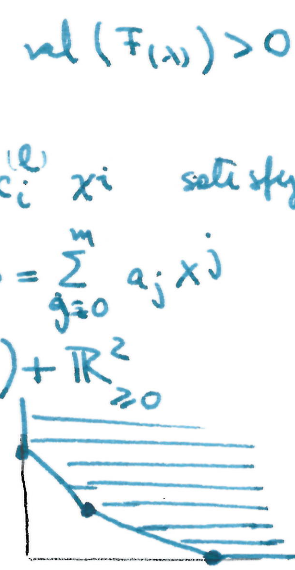
For $G \in K[x]$ define the Newton diagram: $\quad G = \sum_{j=0}^{m} a_j x^j$

$$\mathcal{N}(G) = \operatorname{conv\,hull}(\{ (j, \operatorname{val}(a_j)) \mid a_j \neq 0 \}) + \mathbb{R}^2_{\geq 0}$$

Eg. $G = t^2 + t x + x^3$:



$\rightsquigarrow$



$= \mathcal{N}(G)$

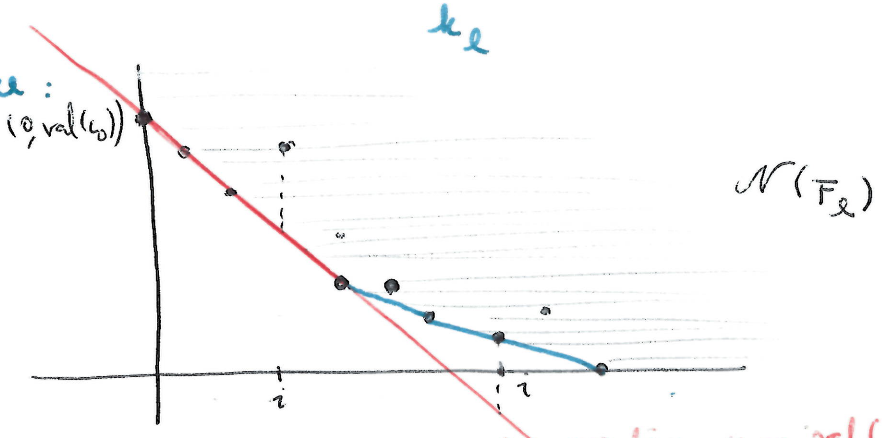
By (1): $\mathcal{N}(F_\ell) \subseteq \mathbb{R}^2_{\geq 0}$

By (4) $\mathcal{N}(F_\ell)$ has a vertex $v_0 = (0, \underbrace{\operatorname{val}(c_0^{(\ell)})}_{>0})$ & by (2) edge from v_0 has negative slope.

Edge connecting v_0 to $(k_\ell, val(c^{(\ell)}_{k_\ell}))$ has slope

$$w_\ell := \frac{val(c_0^{(\ell)}) - val(c^{(\ell)}_{k_\ell})}{k_\ell} < 0$$

Picture:



L $\to$ line $y = val(c^{(\ell)}_{k_\ell}) - w_\ell(x - k_\ell) = val(c_0^{(\ell)}) - w_\ell x$

$\rightsquigarrow G_\ell = t^{-val(c_0^{(\ell)})} F_\ell(t^{w_\ell} x) = \sum_{i=0}^{n} c_i t^{w_\ell \cdot i - val(c_0^{(\ell)})} x^i \in K[x]$

& $val(c_i^{(\ell)} t^{w_\ell\cdot i - val(c_0^{(\ell)})}) = val(c_i^{(\ell)}) + w_\ell\cdot i - val(c_0^{(\ell)}) = val(c_i^{(\ell)}) - (val(c_0^{(\ell)}) - w_\ell i)$ ≥ 0 & $=0$ for $i=0, k_\ell$

because $(i, val(c_i^{(\ell)}))$ is above the line L.

So $G_\ell \in \mathcal{O}_v[X]$ $\rightsquigarrow$ take quotient $f_\ell := \overline{G_\ell} \in \mathbb{K}_v[x]$

• Properties of f_ℓ:

(1) $\deg f_\ell = k_\ell > 0$ (for $i > k_\ell$, coeff of x^i in G_ℓ lies in $\mathfrak{m}_v$)

(2) constant term of $G_\ell = c_0^{(\ell)} t^{-val(c_0^{\ell})}$ has val $=0$ so $f_\ell(0) \neq 0$.

Use $\overline{\mathbb{K}}_v = \mathbb{K}_v$ to pick a root λ_ℓ of f_ℓ in $\mathbb{K}_v$. Write $r_{\ell+1} = mult(f_\ell, \lambda_\ell) \leq k_\ell$

so $f_\ell = (x-\lambda_\ell)^{r_{\ell+1}} g_\ell(x)$ for some $g_\ell \in \mathbb{K}(x)$ & $g_\ell(\lambda_\ell) \neq 0$.

Note: For any lift of λ_ℓ to K: $val(\tilde{\lambda}_\ell) = 0$

• Use λ_ℓ to build $F_{\ell+1}(x)$ as follows:

$$F_{\ell+1}(x) = \sum_{j=0}^{n} c_j^{(\ell+1)} x^j := t^{-val(c_0^{(\ell)})} F_\ell(t^{w_\ell}(x+\lambda_\ell))$$

$$= \sum_{j=0}^{n} \underbrace{\Big[\sum_{i=j}^{n} c_i^{(\ell)} t^{i w_\ell - val(c_0^{(\ell)})} \binom{i}{j} \lambda_\ell^{i-j} \Big]}_{=: c_j^{(\ell+1)}} x^j \quad (**)$$

Claim: $F_{\ell+1}$ satisfies (1) & (2)

Pf/(1): $val(c_j^{(\ell+1)}) \geq \min_{j \leq i \leq n} \Big\{ \underbrace{(val(c_i^{(\ell)}) - (val(c_0^{(\ell)}) - i w_\ell))}_{\substack{>0 \text{ for } i > k_\ell \\ \geq 0 \text{ in general}}} + \underbrace{val\binom{i}{j}}_{\substack{\geq 0 \\ (\text{in } K)}} + (i-j) \underbrace{val(\lambda_\ell)}_{=0} \Big\}$ so ≥ 0 $\forall j$ & >0 for $j > k_\ell$

(2) When considering $\overline{F_{\ell+1}}$ in $\mathbb{K}[x]$ we get that $\deg(\overline{F_{\ell+1}}) \leq k_\ell$

because $c_{k_\ell}^{(\ell+1)} = (c_{k_\ell}^{(\ell)} t^{k_\ell w_\ell - val(c_0^{(\ell)})} \binom{i}{j} \lambda_\ell^{i-j}) + \mathfrak{m}_v$ & the first summand has val $=0$.

So in particular $\mathrm{val}(c_{k_\ell}^{(\ell+1)}) = 0$ ✓ □

Obs: Notice the following identities:

$$F_\ell(t^{w_\ell}x)\, t^{-\mathrm{val}(c_0^{\ell})} = f_\ell(x) + h_\ell(x) \quad \text{for some } h_\ell \in \mathfrak{M}[x].$$

$$\Rightarrow \frac{\partial f_\ell}{\partial x^j}(x) \equiv t^{w_\ell j - \mathrm{val}(c_0^{(\ell)})} \frac{\partial F_\ell}{\partial x^j}\Big|_{(t^{w_\ell}x)} \quad \text{modulo } \mathfrak{M}\,\mathcal{O}_v[x].$$

Next: $$\frac{\partial F_\ell}{\partial x^j}(y) = j! \sum_{i=j}^{n} c_i^{(\ell)} \binom{i}{j} y^{i-j}$$

$$\Rightarrow t^{w_\ell j - \mathrm{val}(c_0^{(\ell)})} \frac{\partial F_\ell}{\partial x^j}\Big|_{t^{w_\ell}x} = j! \sum_{i=j}^{n} c_i^{(\ell)} \binom{i}{j} t^{-(\mathrm{val}\, c_0^{(\ell)} - w_\ell i)} x^{i-j}$$

We specialize at $x = \lambda_\ell$ & set

$$\frac{\partial f_\ell}{\partial x^j}(\lambda_\ell) \equiv j! \sum_{i=j}^{n} c_i^{(\ell)} \binom{i}{j} t^{-(\mathrm{val}\, c_0^{(\ell)} - w_\ell i)} \lambda_\ell^{i-j} \equiv j!\, c_j^{(\ell+1)} \mod \mathfrak{M}$$

Since char $\mathbb{K} = 0$, we conclude:

$$\overline{c_j^{\ell+1}} \equiv \frac{1}{j!}\frac{\partial f_\ell}{\partial x^j}(\lambda_\ell) = \begin{cases} 0 & 0 \leq j < r_{\ell+1} \\ \neq 0 & j = r_{\ell+1} \end{cases} \quad (*)$$

We want $F_{\ell+1}$ to satisfy (3) & (4):

3) If $c_0^{(\ell+1)} = 0$, then $x = 0$ is a root of $F_{\ell+1} \Rightarrow \lambda_\ell t^{w_\ell}$ is a root of F_ℓ.

Since $F_\ell(x) = t^{-\mathrm{val}\, c_0^{\ell-1}} F_{\ell-1}(T^{w_{\ell-1}}(x+\lambda_{\ell-1})) \Rightarrow t^{w_{\ell-1}} t^{w_\ell}\lambda_\ell + t^{w_{\ell-1}}\lambda_{\ell-1}$ is a root of $F_{\ell-1}$

F_ℓ satisfied (3).

By induction, we conclude that $\lambda = \sum_{j=0}^{\ell} t^{w_0 + \cdots + w_j}\lambda_j$ is a root of F_0 as we wanted to find.

So we may assume $c_0^{(\ell+1)} \neq 0$ (ie $F_{\ell+1}$ satisfies (3))

(4) since $c_0^{(\ell+1)} \neq 0$ & $c_0^{(\ell+1)} \in \mathfrak{M}_v$, we conclude $\mathrm{val}(c_j^{(\ell+1)}) > 0$, so (4) holds.

Since $c_j^{(\ell+1)} \in \mathfrak{M}$ $\forall j > k_\ell$ & $(*)$ holds, we conclude $\boxed{k_{\ell+1} \leq r_{\ell+1} \leq k_\ell}$

Here, $k_{\ell+1}$ is the value of the first index with $c_j^{(\ell+1)} \notin \mathfrak{M}_v$

We construct the chain $n \geq k_1 \geq k_2 \geq k_3 \geq \cdots\cdots \geq 0$, so after a certain number of iterations (say m), we have $\boxed{k_\ell = k \quad \forall \ell \geq m}$ so $\boxed{r_\ell = k = k_\ell \,\forall \ell \geq m}$

We conclude mult$(f_\ell, \lambda_\ell) = \deg f_\ell$ & $f_\ell = \mu_\ell (x-\lambda_\ell)^k$ for $\mu_\ell \in \mathbb{K}$ for all $\ell > m$.

• **NEXT**: Find N where $c_j^\ell \in \mathbb{K}(t^{\frac{1}{N}})$ $\forall j$ $\forall \ell$

given ℓ: pick $N_\ell \in \mathbb{Z}_{>0}$ with $c_j^\ell \in \mathbb{K}(t^{\frac{1}{N_\ell}})$ $\forall 0 \le j \le n$.
For example, using the expression (**) defining $c_j^{\ell+1}$, given N_ℓ & w_ℓ, we can pick

$$N_{\ell+1} := \operatorname{lcd}\{N_\ell, w_\ell\}$$

Claim: $N_{\ell+1} = N_\ell$ for $\ell > m$. It suffices to show $w_\ell \in \frac{1}{N_\ell}\mathbb{Z}$ for $\ell > m$

Proof: We claim by induction that $c_i^{(\ell)} \equiv (-1)^{k-i}\binom{k}{i} c_k^{(\ell)} t^{(k-i)w} \lambda_\ell^{k-i}$ mod $\mathfrak{M}$ for $0 \le i \le k$.

Write $F_{\ell+1}(x) = t^{-\mathrm{val}(c_0^{(\ell)})} F_\ell(t^{w_\ell}(x+\lambda_\ell)) \equiv f_\ell(x+\lambda_\ell) = \mu_\ell x^k$

$$\equiv \sum_{j=0}^{k} c_j^{(\ell+1)} x^j$$

(coeff of x^j for $j>k$ lie in $\mathfrak{M}$)

so (***)

$$\boxed{c_j^{(\ell+1)} = \sum_{i=j}^{k} c_i^{(\ell)} t^{iw_\ell - \mathrm{val}(c_0^{(\ell)})} \binom{i}{j} \lambda_\ell^{i-j} \equiv \mu_\ell \delta_{j,k} \bmod \mathfrak{M}}$$

• We start with $i=k$: $c_k^{(\ell)} \equiv 1$ by def, so the claim follows ✓

• Induction step: if true for j, then true for $j-1$:
Write $\tilde{j} = k-j$ & use the identity (***) $/\mu_k$

$$0 \equiv c_{k-j-1}^{(\ell)} t^{(k-j-1)w_\ell} \binom{k-j-1}{k-j-1} \lambda_\ell^{i-(k-j-1)} + \sum_{i=k-j}^{k} (-1)^{k-i} \binom{k}{i} c_k^{(\ell)} t^{(k-i)w_\ell} \lambda_\ell^{k-i} \cdot t^{iw_\ell} \binom{i}{k-j-1} \lambda_\ell^{i-(k-j-1)}$$

$$\equiv \underline{\qquad\qquad\qquad} + \left\{ \sum_{i=k-j}^{k} (-1)^{k-i} \binom{k}{i}\binom{i}{k-j-1} \right\} t^{kw_\ell} \lambda_\ell^{j+1} c_k^{(\ell)}$$

Write

$$\sum_{i=k-j}^{k} (-1)^{k-i} \underbrace{\binom{k}{i}\binom{i}{k-j-1}} = \sum_{i=k-j}^{k} (-1)^{i} \binom{k}{j+1}\binom{j+1}{k-i} = \sum_{i'=0}^{j} (-1)^{k-i'} \binom{k}{j+1}\binom{j+1}{i'} =$$

$$= \binom{k}{j+1}(-1)^k \Big(\underbrace{\sum_{i=0}^{j+1} (-1)^i \binom{j+1}{i}}_{=0} - (-1)^{j+1} \Big)$$

$$= \binom{k}{j+1}(-1)^{k-j}$$

$$\binom{k}{i}\binom{i}{k-j-1} = \frac{k!}{i!\,(k-i)!}\,\frac{i!}{(k-j-1)!\,(j+1-(k-i))!} = \binom{k}{j+1}\binom{j+1}{k-i}$$

So $0 = c_{k-j-1}^{(\ell)} t^{(k-j-1)w_\ell} + (-1)^{k-j}\binom{k}{j+1} t^{kw_\ell} \lambda_\ell^{j+1} c_k^{(\ell)}$.

Then $c_{k-j-1}^{(\ell)} = (-1)^{k-j-1}\binom{k}{j+1} t^{(j+1)w_\ell} \lambda_\ell^{j+1} c_k^{(\ell)}$ as we wanted to show □

Using this, we have $c_{k-1}^{(\ell)} = -\binom{k}{k-1} c_k^{(\ell)} t^{w_\ell} \lambda_\ell \Rightarrow \underset{\boxed{i=0}}{\binom{k}{k-i}} c_0^{(\ell)} \equiv (-1)^{k-1} c_{k-1}^{(\ell)} t^{(k-1)w_\ell} \lambda_\ell^{k-1}$

Take val, $\mathrm{val}(c_0^{(\ell)}) = \mathrm{val}(c_{k-1}^{(\ell)}) + (k-1)w_\ell = \frac{a}{N_\ell} + (k-1)w_\ell$ for $a \in \mathbb{Z}$ [18]

But $w_\ell = \dfrac{\mathrm{val}(c_0^\ell) - \mathrm{val}(c_k^\ell)}{k} = \dfrac{\mathrm{val}(c_0^\ell)}{k}$ because $\mathrm{val}(c_k^\ell) = 0$ for $\ell > m$

$$\Rightarrow \quad k w_\ell = \frac{a}{N_\ell} + (k-1) w_\ell \quad \Rightarrow \quad \boxed{w_\ell = \frac{a}{N_\ell} \in \frac{1}{N_\ell}\mathbb{Z}}$$ as we wanted.

Define $y_\ell := \sum_{j=0}^{\ell} \lambda_j t^{w_0 + \cdots + w_j} \in k((t^{1/N_\ell}))$

Using our claim: $y := \sum_{j \geq 0} \lambda_j t^{w_0 + \cdots + w_j} \in k((t^{1/N}))$ $[N = N_m]$

We claim that $F(y) = 0$.

<u>Obs</u>: $z_i = \sum_{j \geq i} \lambda_j t^{w_i + \cdots + w_j} = \dfrac{y - y_{i-1}}{\prod_{0 \leq j \leq i-1} t^{w_j}}$ for all $i > 0$. & $y = z_0$

so $z_i = t^{w_i}(z_{i+1} + \lambda_i)$ for all i

In particular: $F_{\ell+1}(z_{\ell+1}) = t^{-\mathrm{val}(c_0^{(\ell)})} F_\ell(t^{w_\ell}(z_{\ell+1} + \lambda_\ell)) = t^{-\mathrm{val}(c_0^{(\ell)})} F_\ell(z_\ell)$

$$F_{(y)} = F_0(y) = F_0(z_0) = t^{\mathrm{val}(c_0^{(0)})} F_1(z_1) \underset{\text{induction}}{=} t^{\sum_{i=0}^{\ell} \mathrm{val}(c_0^{(i)})} F_{\ell+1}(z_{\ell+1})$$

Note $z_{\ell+1} \in \mathcal{O}_r((t^{1/N}))$ & $F_{\ell+1} \in \mathcal{O}_r[x]$ so $\mathrm{val}(F_{\ell+1}(z_{\ell+1})) \geq 0$

Then $\mathrm{val}(F_{(y)}) \geq \sum_{i=0}^{\ell} \mathrm{val}(c_0^{(i)}) + \mathrm{val}(F_{\ell+1}(z_{\ell+1})) \geq \sum_{i=0}^{\ell} \underbrace{\mathrm{val}(c_0^{(i)})}_{\in \frac{1}{N}\mathbb{Z}_{>0}}$

So we conclude $\mathrm{val}(F_{(y)}) = +\infty$, so $F_{(y)} = 0$, as desired. $\square$