Lecture III: Polyhedral geometry

GOALS:
- Basic definitions / examples of polyhedra & polyhedral complexes
- Show what can we say about algebraic geometry via discretization (on the polyhedral side)

2 polyhedral objects $\rightarrow$ tropical varieties
$\searrow$ toric varieties = equivariant compactifications X of algebraic tori $T \simeq (k^*)^n$ over field k
$\hookrightarrow$ action of $T \circlearrowright T$ extends to $T \circlearrowright X$ in a natural & compatible way

Given $T \rightsquigarrow$ character lattice $M := \operatorname{Hom}(T, k^*) \simeq \mathbb{Z}^n$ $\longrightarrow$ polynomials side

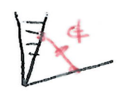
cocharacter " $N = M^\vee := \operatorname{Hom}(M, \mathbb{Z}) \simeq \mathbb{Z}^n$ $\longrightarrow$ fans side.

Dictionary: $\left\{\begin{array}{l}\text{algebraic geometry} \\ \text{of toric varieties}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{combinatorial properties of} \\ \text{• polytopes in } M \\ \text{or • fans in } N = M^\vee\end{array}\right\}$

§1 Basics on convex geometry:

- V vector space / $\mathbb{R}$. $C \subseteq V$ is called **convex** if for every $v_1, v_2 \in C$, we have $\lambda v_1 + (1-\lambda) v_2 \in C \quad \forall\, 0 \leq \lambda \leq 1$
- C is called a **cone** if $v \in C \Rightarrow \lambda v \in C \quad \forall \lambda \geq 0$

Our favorite objects: convex cones

BAD $\rightsquigarrow$  = don't union of 2 convex cones

- The **convex hull** $\operatorname{conv}(U)$ of a set $U \subseteq V$ is the smallest set containing U

Why? V is convex, V_1, V_2 convex $\Rightarrow V_1 \cap V_2$ is convex.
If U = finite set, $\operatorname{conv}(U)$ is a **polytope**. Minimal pts needed to "span" = vertices $\rightsquigarrow$ [V-rep-n].

- A **convex polyhedral cone** is a cone generated by finitely many vectors, ie

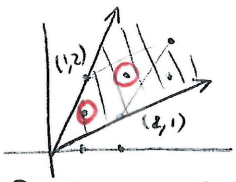
$$C_{(v_1,\ldots,v_n)} := C = \{\lambda_1 v_1 + \cdots + \lambda_n v_n \mid \lambda_i \geq 0 \ \forall i\} \quad \text{for } v_1, \ldots, v_n \in V.$$

Notation: $C = \mathbb{R}_{\geq 0}\langle v_1, \ldots, v_n\rangle$ [V-representation]

Nice properties: $V = N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ ($N \cong \mathbb{Z}^d$ natural lattice, $V \simeq \mathbb{R}^n$)

- C is **simplicial** if $v_1, \ldots, v_n$ are linearly independent
- C is **smooth** if its minimal generators (w_i st $\mathbb{R}_{\geq 0}\langle v_i\rangle \cap \mathbb{Z}^n = \mathbb{Z}_{\geq 0}\langle w_i\rangle$) form part of a $\mathbb{Z}$-basis of N.

Eg in $\mathbb{R}^2$:



$C = \mathbb{R}_{\geq 0}\langle (1,2), (2,1)\rangle$

simplicial, not smooth

$\left|\det\begin{vmatrix} 1 & 2 \\ 2 & 1\end{vmatrix}\right| = |1-4| = |-3| = 3 \neq 1$

$\exists\, 2 = 3-1$ lattice pts in fund. domain 

Eg in $\mathbb{R}^3$

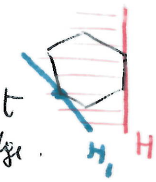
not simplicial

cone sec" 

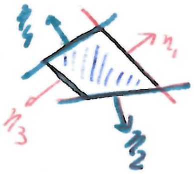
$\left\{\begin{array}{l} v_1 = (1,0,1) \\ v_2 = (0,1,1) \\ v_3 = (0,0,1) \\ v_4 = (1,1,1)\end{array}\right.$

Alternative description = H-representation (of convex set) : $C = \{x \in V \simeq \mathbb{R}^n : A \cdot x \leq b\}$ $A \in \mathbb{R}^{m \times n}$

Def: Let K be a convex set in V. A supporting hyperplane is an affine hyperplane H st $H \cap K \neq \emptyset$ & K is contained in one of the 2 halfspaces determined by H



$\rightsquigarrow$ Rows of A = supporting hyperplanes



$\eta_i \cdot x \leq b_i$

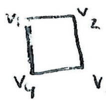
When C is cone : $b = \underline{0}$.

Algorithm for polytopes (= convex hulls of finitely many pts) = Fourier-Motzkin (see Ziegler's "Polytopes")

• Polyhedron = describe by H-representations

Def: A face of K is $K \cap H$ for H a supporting hyperplane.

Exercise: If $C = \mathbb{R}_{\geq 0}\langle v_1, \ldots, v_n\rangle$, then a face of C is generated by a subset of $\{v_1, \ldots, v_n\}$ (NOT any!)



$\{v_1, v_3\}$ does not generate a face....

• A facet is a face of codim 1.

NOTATION: If σ is a convex polyhedral cone, τ a face of σ, write $\tau \lneq \sigma$

$\rightsquigarrow$ face poset of σ

Note: If $H = \{\omega \cdot x = b\}$ $\Rightarrow$ face $K \cap H = \text{face}_\omega K = \{x \in K : \omega \cdot y \leq \omega \cdot x \ \forall y \in K$

§2 Collections of cones/polyhedra:

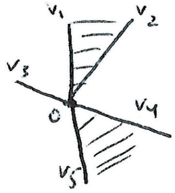
Object of interest: collections w/ nice intersections among constituent objects.

Def 1: A polyhedral fan in V is a collection $\Sigma = \{\sigma\}_{\sigma = \text{polyhedral cone}}$ s.t.

① If $\tau \lneq \sigma \Rightarrow \tau \in \Sigma$ (closed under "faces")

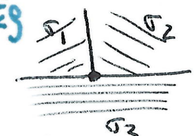
② $\sigma_1, \sigma_2 \in \Sigma$, then $\sigma_1 \cap \sigma_2$ is a face of σ_i for $i = 1, 2$. (Note $\sigma_1 \cap \sigma_2 \ni 0$)

Eg:



$\Sigma = \{0, \langle v_1\rangle, \langle v_2\rangle, \langle v_3\rangle, \langle v_4\rangle, \langle v_5\rangle, \langle v_1, v_2\rangle, \langle v_4, v_5\rangle\}$

Non-Eg



$\sigma_1 \cap \sigma_3 \not\leq \sigma_3$

Fundamental for T.V. / tropicalizations over trivially valued fields

Def 2 A polyhedral complex is a collection $\Sigma = \{P \mid P \text{ polyhedron}\}$ s.t.

① If $Q \leq P \Rightarrow Q \in \Sigma$

② $P, Q \in \Sigma \Rightarrow P \cap Q = \emptyset$ or $(P \cap Q \leq P$ & $P \cap Q \leq Q)$

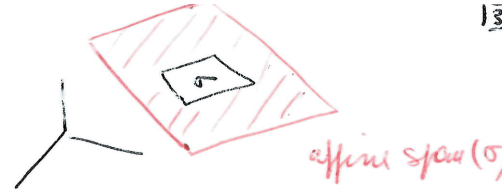
non pure, dim = 2, $f = (4, 9, 3)$

Elements of Σ = cells

Def 3: $\text{supp}(\Sigma) = |\Sigma| = \{x \in V : x \in P$ for some $P \in \Sigma\}$ Support of the complex

- Combinatorial properties of Σ:

  - dimension of a cell $=$ dim affine span of the cell



  - $\dim \Sigma = \max_{\sigma \in \Sigma} \{ \dim(\sigma) \}$
  - Σ is pure if all maximal cells of Σ have the same dimension (order on cells = inclusion as faces)
  - f-vector $= (f_0, \dots, f_N)$ $\quad N = \dim \Sigma \quad$ $f_i := \#\{\text{cells of dimension } i \text{ in } \Sigma\}$

- Useful tools: refinements & Minkowski sums:

Def 1 $\Sigma_1, \Sigma_2 \subseteq V$ polyhedral complexes. The common refinement of Σ_1 & Σ_2 is the polyhedral complex $\Sigma_1 \wedge \Sigma_2 = \{ P \cap Q : P \in \Sigma_1, Q \in \Sigma_2 \}$

Note: $|\Sigma_1 \wedge \Sigma_2| = |\Sigma_1| \cap |\Sigma_2|$

Eg:

Def 2: $A, B \subseteq V$ polyhedra. The Minkowski sum is

$$A + B = \{ a + b \mid a \in A, b \in B \}$$

FACT: $A+B$ is also a polyhedron

Eg:

$= \mathbb{R}^2_{\geq 0}$