Lecture IV: Polyhedra, subdivisions & tropical hypersurfaces

§1 Polyhedra vs fans:

$N = \mathrm{Hom}((K^*)^n, K^*) \simeq \mathbb{Z}^n$, $M = N^\vee = \mathrm{Hom}(N, \mathbb{Z})$ $\quad M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$

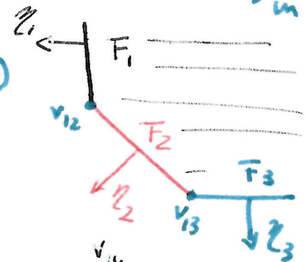
Recall: A polyhedron P in $M_{\mathbb{R}}$ is a finite intersection of sets of the form $\{x : Ax \leq \underline{b}\}$

Def: Given $P \subseteq M_{\mathbb{R}}$ a polyhedron, the normal fan of P is $\mathcal{N}_P = \{\mathcal{N}_P(F)\}_{F \leq P}$

$$\mathcal{N}_P(F) = \text{closure}\,(\{\omega \in N_{\mathbb{R}} : \underbrace{\mathrm{face}_\omega(P)}_{= P \cap H_\omega} = F\}) \subseteq N_{\mathbb{R}} = \mathbb{R}^n$$

↳ in Euclidean top.

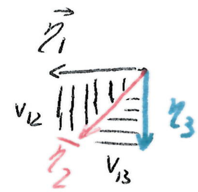
Eg: ①



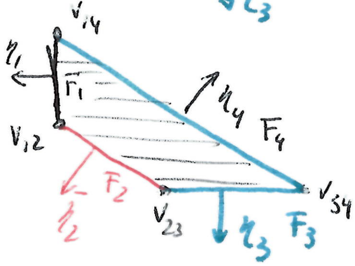
$\mathcal{N}_P(F_i) = \mathbb{R}_{\geq 0}(\vec{\eta}_i)$

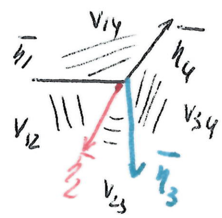
$\mathcal{N}_P(v_{ij}) = \mathbb{R}_{\geq 0}\langle \vec{\eta}_i, \vec{\eta}_j \rangle$

$\mathcal{N}_P(P) = \{0\}$



②





Remarks: • normals = OUTER normals ⇝ outer normal fan

• Analogously we can work w/ inner normals ⇝ inner normal fan

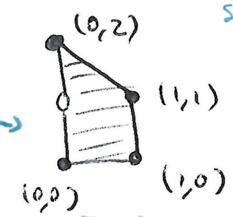
• P bounded $\Longrightarrow |\mathcal{N}_P| = N_{\mathbb{R}}$

• From Laurent polynomials in $S = K[x_1^{\pm}, \ldots, x_n^{\pm}]$ to polyhedra: $\quad N = \mathbb{Z}^n \leadsto \{x_1, \ldots, x_n$ basis$\}$

Def: Given $f = \sum_{\substack{u \in \mathbb{Z}^n \\ \text{finite}}} c_u x^u \in S$, the Newton polytope of f = $NP(f)$

is $NP(f) = \text{conv hull}\,\{u \mid c_u \neq 0\}$ in $M_{\mathbb{R}}$. $\quad$ $\mathrm{supp}(f) = \{u : c_u \neq 0\}$ is finite.

Eg: $K = \mathbb{C}\{\{t\}\}$ $\qquad f = 1 + t^{-1}x + y^2 - txy$ ⇝



Prop: • $\mathcal{N}_{P+Q} = \mathcal{N}_P \wedge \mathcal{N}_Q$ (Minkowski sum vs Common Refinements)

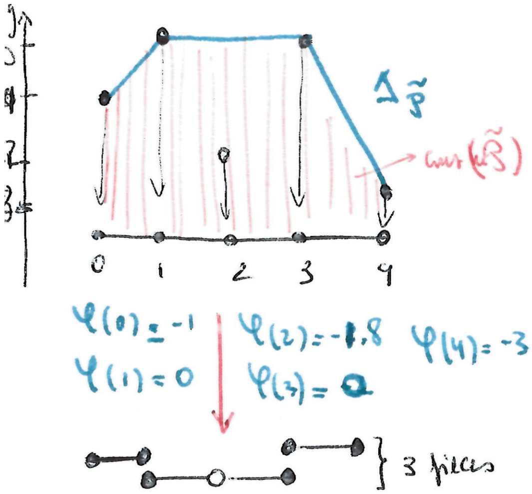
• $NP(fg) = NP(f) + NP(g)$

§2 Regular (coherent) subdivisions: Fix $M = \mathbb{Z}^n$

Idea: $\underbrace{P}_{\text{vertices in } \mathbb{Z}^n}$ lattice polyhedron in $\mathbb{R}^n \longleftrightarrow \mathbb{R}^{n+1}$ by assigning heights to its lattice pts using the function $\varphi : P \cap \mathbb{Z}^n \longrightarrow \mathbb{R} \cup \{\infty\}$

We consider the set $\tilde{\mathcal{P}} = \{(v, y) \in \mathbb{R}^n \times \mathbb{R} \mid v \in P \cap \mathbb{Z}^n, \ \varphi(v) \geq y\}$

Eg: $\mathcal{P} = \{0, 1, 2, 3, 4\}$



$\varphi(0) = -1$ $\varphi(2) = -1.8$ $\varphi(4) = -3$
$\varphi(1) = 0$ $\varphi(3) = 0$

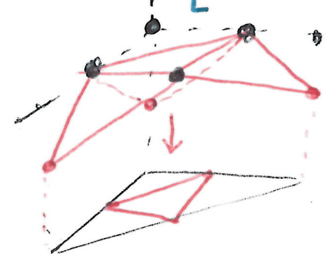
then, we take $\mathrm{conv}(\tilde{\mathcal{P}})$ & the upper convex hull $\Delta_{\tilde{\mathcal{P}}}$:

$$\Delta_{\tilde{\mathcal{P}}} = \{(x, y) \in \mathbb{R}^{n+1} \mid (x, y) \in \mathrm{conv}(\tilde{\mathcal{P}}),\ (x, y+\varepsilon) \notin \mathrm{conv}(\tilde{\mathcal{P}}) \text{ for } \varepsilon > 0\}$$

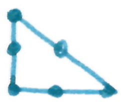
- The upper facets of $\Delta_{\tilde{\mathcal{P}}}$ are the faces of $\mathrm{conv}\, \tilde{\mathcal{P}}$ with outer normal $\vec{\eta} = \langle x, y \rangle$ with $y \geq 0$
- Take upper facets of $\Delta_{\tilde{\mathcal{P}}}$ & project from $\Delta_{\tilde{\mathcal{P}}}$ back to $\mathcal{P}$

$\Rightarrow$ $\mathcal{P}$ gets subdivided into polyhedra &

The result is the regular subdivision of $\mathcal{P}$ induced by φ. [marked subdivision: $p \in P \cap \mathbb{Z}^n$ is marked if $(p, \varphi(p)) \in$ upper facets of $\Delta_{\tilde{\mathcal{P}}}$]

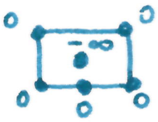


Ex: $\mathcal{P} =$

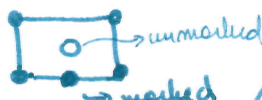


$\varphi(1,1) = \varphi(1,0) = \varphi(0,1) = 0$
$\varphi(1,0) = \varphi(0,1) = \varphi(0,0) = -1$

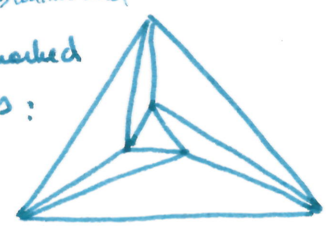
Ex 2



$\rightsquigarrow$ subdivision



Note: There are non-regular subdivisions:



§3 Tropical hypersurfaces:

Fix K valued field, $f \in K[x_1^{\pm}, \dots, x_n^{\pm}]$ $\rightsquigarrow$ $P = NP(f)$ can use valuation of the coefficients of f to define a height function:

$$f = \sum_{u \in \mathbb{Z}^n} c_u x^u = \sum_{u \in P \cap \mathbb{Z}^n} c_u x^u \rightsquigarrow \boxed{\varphi(u) = -\mathrm{val}(c_u)}$$

- We define the Newton subdivision of f to be the regular subdivision of $NP(f)$ induced by φ.
- We define the tropicalization of f: $\mathrm{trop}(f): \mathbb{R}^n \longrightarrow \mathbb{R}$ to be the function

$$\mathrm{trop}(f)(w) = \bigoplus_u (-\mathrm{val}\, c_u) \odot w^{\odot u} = \max_{u \in \mathbb{Z}^n} \{-\mathrm{val}(c_u) + \langle w, u \rangle\}$$

Remark: $\mathrm{trop}(f)$ is piecewise linear & convex function [If min convention, use $\mathrm{val}(c_u)$ instead & lower convex hull]

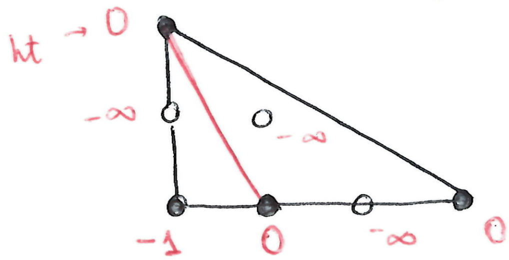
Definition: The tropical hypersurface $\mathcal{T}(V(f)) \subseteq \mathbb{R}^n$ is the set

$\mathcal{T}(V(f)) = \{w \in \mathbb{R}^n :$ the max in $\mathrm{trop}(f)(w)$ is achieved at least twice$\} =: V(\mathrm{trop}(f))$

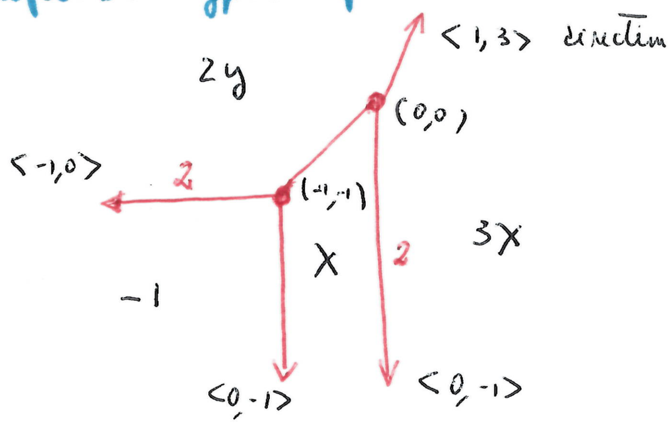
Example: $f = y^2 - x^3 + x + t$ Elliptic plane cubic / $\mathbb{C}\{\{t\}\}$

- $\text{trop}(f)_{(X,Y)} = \max\{2Y, 3X, X, -1\}$

- Newton subdivision of f =



- Tropical hypersurface: dual to the subdivision:



weights ⟺ # components of initial deg.
"
(number of lattice pt − 1) in the dual edge of the N. subdiv

Next Time = alternative characterization that justifies the picture.