Lecture V: Tropical Hypersurfaces & The Fundamental Theorem

Recall of $f = \sum_{\substack{u\in\mathbb{Z}^n\\ \text{finite}}} c_u\, x^u \in K[\underline{x}^{\pm}] \rightsquigarrow \operatorname{trop}(f)(\underline{\omega}) = \bigoplus_{\substack{u\in\mathbb{Z}^n\\ \text{finite}}} (-\operatorname{val}(c_u)) \odot \underline{\omega}^{\odot u} : \mathbb{R}^n \to \mathbb{R}$, $\underline{\omega} \in \mathbb{R}^n$

(K, val) valued field, $\overline{K} = K$

$= \max_{u\in\mathbb{Z}^n} \{-\operatorname{val}(c_u) + \langle \underline{\omega}, u\rangle\}$

Def The tropical hypersurface $\mathcal{T}(V(f)) \subseteq \mathbb{R}^n$ is the set:

$\{\omega \in \mathbb{R}^n : \text{the max in } \operatorname{trop}(f)(\omega) \text{ is achieved at least twice}\} = V(\operatorname{trop} f)$

GOAL: Given an alternative characterization & show the duality w/ Newton subdivision of f.

§1 Kapranov's Theorem:

To give an alternative characterization of $\mathcal{T}(V(f))$, we need a splitting of val

Assume: $1 \in \Gamma_{\operatorname{val}} \subseteq \mathbb{R}$

Lemma: Assume $K = \overline{K}$, then $\operatorname{val}: K^\times \twoheadrightarrow \Gamma_{\operatorname{val}}$ splits, ie $\exists\, \Psi : (\Gamma_{\operatorname{val}}, +) \to (K^\times, \cdot)$ homomorphism with $\operatorname{val} \circ \Psi(\omega) = \omega \ \ \forall\, \omega \in \Gamma_{\operatorname{val}}$. $\to$ [see end of notes]

Pf/ • For $a \in K^\times$ & $n \in \mathbb{N}$ $\exists\, a^{1/n} \in K^\times$ (because $(K^\times, \cdot)$ is divisible) $\overset{(*)}{\rightsquigarrow} \exists\, \Psi : (\mathbb{Q}, +) \to (K^\times, \cdot)$ with $\Psi(1) = a$ (see end of the lecture notes $= \Psi$ is homom.)

$1 \in \Gamma_{\operatorname{val}}$

• From Lecture II $(\Gamma_{\operatorname{val}}, +)$ divisible abelian gps

• $\Gamma_{\operatorname{val}} \subseteq \mathbb{R}$ additive subgroup $\Rightarrow$ torsion free

$\Rightarrow \Gamma_{\operatorname{val}} \underset{\alpha}{\overset{\simeq}{\leftarrow}} \bigoplus_I \mathbb{Q}$ (I maybe uncountable) [pick a basis of $\Gamma_{\operatorname{val}}$ as a $\mathbb{Q}$ vs]

On each summand, we get $1 \xrightarrow{\alpha} \omega_i \in \Gamma_{\operatorname{val}} \rightsquigarrow$ Find $a_i \in K^\times$ with $\operatorname{val}(a_i) = \omega_i$

$\rightsquigarrow \Psi_{\omega_i} = \Psi_i : (\mathbb{Q}, +) \longrightarrow (K^\times, \cdot)$ sending 1 to a_i. By universality of $\bigoplus_I$,

then $\Psi = \bigoplus_{i\in I} \Psi_i$ is a section to val. □

Notation. Borrowing the natural splitting from $\mathbb{K}\{\{t\}\}$, we denote $t^\gamma := \Psi(\gamma) \in K^\times$ for $\gamma \in \Gamma_{\operatorname{val}}$

Definition: Given a splitting $\gamma \mapsto t^\gamma$ of val & $\omega \in \mathbb{R}^n$ we define the ω-initial form of f as

$$\operatorname{in}_\omega(f) = \sum_{\substack{u\\ -\operatorname{val}(c_u) + \langle \omega, u\rangle = \operatorname{trop}(f)(\omega)}} \overline{t^{-\operatorname{val}(c_u)} c_u}\, x^u \in \mathbb{k}[x_1^{\pm}, \ldots, x_n^{\pm}]$$

Note: $t^{-\operatorname{val}(c_u)} c_u$ has $\operatorname{val} = 0$, so its class in $\mathcal{O}_v/\mathfrak{m}_v$ is $\neq 0$.

• Alternatively: for $\omega \in \Gamma_{\operatorname{val}}^n$: $\operatorname{in}_\omega(f) = \sum_u \overline{t^{\operatorname{trop}(f)(\omega) - \langle \omega, u\rangle} c_u}\, x^u \in \mathbb{k}[x_1^{\pm}, \ldots, x_n^{\pm}]$

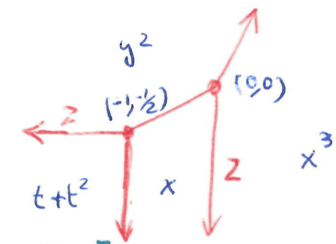
Note $\operatorname{val}(t^{\operatorname{trop}(f)(\omega) - \langle \omega, u\rangle} c_u) = \operatorname{trop}(f)(\omega) - \langle \omega, u\rangle - \operatorname{val}(c_u) \geq 0 \ \ \forall u$

& $= 0$ iff u realizes the max defining $\operatorname{trop}(f)(\omega)$

In particular, — in $\mathcal{O}_v/\mathfrak{m}_v$ is well defined & $\neq 0$ over those u realizing $\operatorname{trop}(f)(\omega)$

Example (last time) $f = y^2 - x^3 + x + (t+t^2)$ $\quad \text{trop}(f)_{(X,Y)} = \max\{2Y, 3X, X, -1\}$

X,Y	$\text{Trop}(f)(X,Y)$	monomials	$\text{in}_{(X,Y)}(f)$
$(0,0)$	0	y^2, x^3, x	$y^2 - x^3 + x$
$(1,0)$	3	x^3	$-x^3$ $[m_{\text{Trop}} = 0]$
$(0,-1)$	0	x^3, x	$-x^3 + x$ $[m_{\text{Trop}} = 2]$
$(-2,-2)$	-1	1	$\overline{t^{-1}(t+t^2)} = \overline{1+t} = 1$ $[m_{\text{Trop}} = 0]$
$(-1,\frac{-1}{2})$	-1	$1, x, y^2$	$y^2 + x + \overline{t^{-1}(t+t^2)} = y^2 + x + 1$ $[m_{\text{Trop}} = 1]$



Theorem (Kapranov) Let $\overline{K} = K$ equipped with a non-trivial valuation & a splitting of val. Let $f \in K[x_1^{\pm}, \dots, x_n^{\pm}]$ nonzero. Then, the following 3 sets coincide:

(1) $\mathcal{T}(V(f)) \subseteq \mathbb{R}^n$

(2) $\{ w \in \mathbb{R}^n \mid \text{in}_w(f) \text{ is } \underline{\text{not}} \text{ a monomial} \}$

(3) closure of $\{ \underbrace{(-\text{val}(y_1), \dots, -\text{val}(y_n))}^{=: -\text{val}(\underline{y})} : (y_1, \dots, y_n) \in V(f) \subseteq (K^\times)^n \}$ in $\mathbb{R}^n$

(Here $V(f) = \{ \underline{y} \in (K^\times)^n \mid f(y_1, \dots, y_n) = 0 \}$)

Furthermore, if f is irreducible & $w \in \Gamma_{\text{val}} \cap \mathcal{T}(V(f))$ then

$$(-\text{val})^{-1}(w) = \{ y \in V(f) : -\text{val}(\underline{y}) = w \}$$

is Zariski dense in $V(f)$.

Def $\text{mult}_{\text{Trop}}(w) := \#\{\text{irred comp of } V(\text{in}_w(f)) \subseteq (\Bbbk^\times)^n\}$ counted with multiplicity (future lectures show well def)

Note:
- If K has non-triv valuation but $K \neq \overline{K}$, then work with $L = \overline{K}$ & an extension of val. ($\leq [L:K]_{\text{sep}} < \infty$ of them)
- If K is trivial valuation, take $L = \overline{K}((\mathbb{R}))$ (generalized power series) with t-val.

In both cases $\text{trop}(f)$ viewing $f \in K[x_1^{\pm} \dots x_n^{\pm}]$ or $f \in L[x_1^{\pm}, \dots, x_n^{\pm}]$ is the same.

Example (cont.) Using Kapranov's Thm, $\mathcal{T}(V(f))$ is the complement of ≤ 4 polyhedra (one per monomial in f)

To find (X,Y) with $\text{in}_w(f) = y^2$: $\quad 2Y > 3X, \quad 2Y > X, \quad 2Y > -1$

$\qquad Y > \frac{3X}{2} \qquad Y > \frac{X}{2} \qquad Y > \frac{-1}{2}$

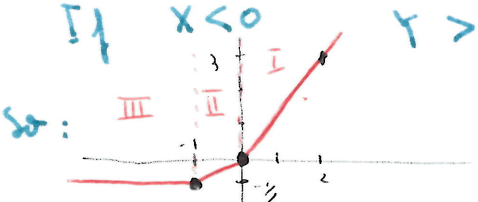
If $X \geq 0$: $\quad Y > \underbrace{\frac{3X}{2}}_{\geq 0}, \ Y > \frac{-1}{2} \iff Y > \frac{3X}{2} \quad X > 0 \quad$ I

If $X < 0$ $\quad Y > \frac{X}{2}, \ Y > \frac{-1}{2}$

$\quad \nearrow \ Y > \frac{-1}{2} \geq \frac{X}{2} \iff Y > \frac{-1}{2}, \ -1 \geq X \quad$ II

$\quad \searrow \ Y > \frac{X}{2} \geq \frac{-1}{2} \iff Y > \frac{X}{2}, \ 0 > X \geq -1 \quad$ III

So:



similarly for the other 3 monomials

Warning: Not every monomial in supp(f) is a w-initial form

Ex: $f = y^2 - x^3 + x + 1$ in $\mathbb{C}[x,y]$.

Claim: x will never be a w-initial form for any $w \in \mathbb{R}^n$.

Reason: $\mathrm{Trop}(f)_{(X,Y)} > 3X, Y, 0$ will never hold.

(1) If $X > 0 \rightsquigarrow 3X > X$.

(2) If $X < 0 \rightsquigarrow 0 > X$

(3) If $X = 0 \rightsquigarrow 3X = X = 0$.

Proof of (*) from Lemma (p.1) Why is $\Psi : (\mathbb{Q}, +) \longrightarrow (K^\times, \cdot)$ with $\Psi(1) = a$ a homomorphism?

Reason: $(K^\times, \cdot)$ is an injective $\mathbb{Z}$-module because $K^\times$ is divisible; meaning $\mathrm{Hom}(-, K^\times)$ is exact

Take $0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$ apply $\mathrm{Hom}(-, K^\times)$ to get

$$0 \to \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, K^\times) \longrightarrow \mathrm{Hom}(\mathbb{Q}, K^\times) \longrightarrow \mathrm{Hom}(\mathbb{Z}, K^\times) \longrightarrow 0 \qquad \underline{\text{ses}}$$

In particular, any group homomorphism $\Psi : (\mathbb{Z}, +) \to (K^\times, \cdot)$ can be lifted (not necessarily uniquely) to a homomorphism $\Psi : (\mathbb{Q}, +) \longrightarrow (K^\times, \cdot)$.

If we define $\Psi : (\mathbb{Z}, +) \longrightarrow (K^\times, \cdot)$ we get $\Psi(1) = a$.

$$k \longmapsto a^k$$

Direct proof: Follow the proof of Baer's Criterion using $K^\times =: I$.

Baer: Fix I abelian gp. Assume I is divisible, ie for all $p \in I$, $n \in \mathbb{Z}_{>0}$:

$$\begin{tikzcd}
1 \in \mathbb{Z} \arrow[r, hook, "\text{mult by } n"] \arrow[d, "\eta_p"'] & \mathbb{Z} \arrow[dl, dashed] \\
p \in I &
\end{tikzcd}$$

$\exists$ a lift of η_p.

Then: $\forall p \in I$

$$\begin{tikzcd}
1 \in \mathbb{Z} \arrow[r, hook] \arrow[d, "\eta_p"'] & \mathbb{Q} \arrow[dl, dashed] \\
p \quad I &
\end{tikzcd}$$

$\exists$ a lift of η_p.

Pf/ Define $\mathcal{E}_p = \{ (E, \varphi) :$

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, hook] \arrow[d, "\eta_p"'] \arrow[dr, phantom, "\circlearrowleft"] & E \arrow[r, hook] \arrow[dl, dashed, "\varphi \text{ gp hom.}"] & \mathbb{Q} \\
I & &
\end{tikzcd}$$

$\}$

- $\mathcal{E}_p$ is non-empty: $(\mathbb{Z}, \eta_p) \in \mathcal{E}_p$.
- Define an order $\leqslant$ on $\mathcal{E}_p$: $(E, \varphi) \leqslant (E', \varphi')$ if $E \hookrightarrow E'$ & $\varphi'|_E = \varphi$.
- Every non-empty chain in $\mathcal{E}_p$ $\{(E_i, \varphi_i)\}_{i \in \Lambda}$ has a maximal element $(\cup E_i, \cup \varphi_i)$ in $\mathcal{E}_p$.

By Zorn's Lemma, $\mathcal{E}_p$ has a maximal element (E, ϕ).

- If $E = \mathbb{Q}$, we are done. ✓
- Otherwise, pick $m \in \mathbb{Q} \setminus E$. Define $\mathfrak{a} \subseteq \mathbb{Z}$ ideal by $\mathfrak{a} = \{r \in \mathbb{Z} : rm \in E\}$

Note $0 \subsetneq \mathfrak{a} \subsetneq \mathbb{Z}$ by construction.

By the divisibility condition on I, we can find a lift of $\mathfrak{a} \simeq \mathbb{Z} \xrightarrow{\cdot m} E \xrightarrow{\Phi} I$.
$r \longmapsto rm$

$$\begin{tikzcd}
r \in \mathfrak{a} \simeq \mathbb{Z} \arrow[r, hook] \arrow[d] & \mathbb{Z} \arrow[ddl, dashed, "\xi"] \\
rm \in E \arrow[d, "\phi"'] \arrow[ur, phantom, "\circlearrowleft"] & \\
I &
\end{tikzcd}$$

ξ a lift of $\mathbb{Z} \longrightarrow I$. (**)

Consider the ab gp $\tilde{E} = \langle E, m \rangle$ (generated by E & $\langle m \rangle$).

We construct $\tilde{\Phi} : \tilde{E} \longrightarrow I$ as $\tilde{\Phi}(e + km) = \phi(e) + k\xi(1)$ for any $k \in \mathbb{Z}$, $e \in E$

Claim: $\tilde{\Phi}$ is well-defined (independent of the expression giving an element in $\tilde{E}$).

If so, $\tilde{\Phi}$ is automatically a gp. homomorphism.

Proof of claim: $e_1 + k_1 m = e_2 + k_2 m \iff \underbrace{e_1 - e_2}_{\in E} = \underbrace{(k_2 - k_1)}_{\in \mathbb{Z}} m$

so $k_2 - k_1 \in \mathfrak{a}$ so $\xi(k_2 - k_1) \underset{\text{by } (**)}{=} \Phi((k_2-k_1)m) = \Phi(e_1 - e_2) = \phi(e_1) - \phi(e_2)$

$k_2\xi(1) - k_1\xi(1) = \xi(k_2) - \xi(k_1) = \xi(k_2-k_1)$

We conclude $\phi(e_1) + k_1\xi(1) = \phi(e_2) + k_2 \xi(1)$.

It follows that

$$\begin{tikzcd}
E \arrow[r, hook] \arrow[d, "\Phi"'] \arrow[dr, phantom, "\circlearrowleft"] & \tilde{E} \arrow[dl, "\tilde{\Phi}"] \\
I &
\end{tikzcd}$$

so $(\tilde{E}, \tilde{\Phi}) \in \mathcal{E}_P$ & $(E, \phi) \precneqq (\tilde{E}, \tilde{\Phi})$ contradicting the maximality of (E, ϕ). $\square$