Lecture VI: The Fundamental Theorem & Structure Theorem for Trop Hypersurfaces

Recall: $0 \neq f = \sum_{u \text{ finite}} c_u x^u \in K[x_1^{\pm}, \dots, x_n^{\pm}]$ & $\underset{\in \Gamma_{val}}{\gamma} \mapsto t^{\gamma}$ splitting of the valuation val on K.

- $\mathrm{in}_\omega(f) = \sum_u \overline{t^{\mathrm{trop}(f)(\omega) - \langle \omega, u\rangle} c_u}\, x^u \in \Bbbk[x_1^{\pm}, \dots, x_n^{\pm}]$ is the ω-initial form of f for $\omega \in \Gamma_{val}^n$, $\Bbbk = \mathcal{O}_v/\mathfrak{m}_v$

Theorem (Kapranov): Assume $K = \overline{K}$ equipped with a non-trivial valuation & a splitting $\gamma \mapsto t^\gamma$ of val. For $0 \neq f$ in $K[x_1^{\pm}, \dots, x_n^{\pm}]$ we have equality between the following sets:

(1) $\mathcal{T}(V(f)) = \{ \omega \in \mathbb{R}^n \mid \mathrm{trop}(f)_{(\omega)}$ is achieved for at least 2 exponents u in $\mathrm{supp}(f)\}$

(2) $\{\omega \in \mathbb{R}^n \mid \mathrm{in}_\omega(f)$ is not a monomial$\}$

(3) closure of $\{(-\mathrm{val}(y_1), \dots, -\mathrm{val}(y_n)) : y$ in $(K^\times)^n$ $f(y) = 0\}$ in $\mathbb{R}^n$.

Furthermore if f is irreducible & $\omega \in \Gamma_{val}^n \cap \mathcal{T}(V(f))$, then
$$(-\mathrm{val})^{-1}(\omega) = \{ y \in V(f) : \mathrm{val}(y) = \omega\}$$
is Zariski dense in $V(f)$.

- **Proof of (1) $\Leftrightarrow$ (2)**:

Pick $\omega \in \mathcal{T}(V(f))$, so $W = \mathrm{trop}(f)_{(\omega)}$ is achieved for at least 2 exponents u, u' in $\mathrm{supp}(f)$. In particular, $t^{W - \langle \omega, u\rangle} c_u \in \mathcal{O}_v^\times$ & $t^{W - \langle \omega, u'\rangle} c_{u'} \in \mathcal{O}_v^\times$ so $\overline{t^{W - \langle \omega, u\rangle} c_u}\, x^u + \overline{t^{W - \langle \omega, u'\rangle} c_{u'}}\, x^{u'}$ is present in $\mathrm{in}_\omega(f)$, so it can't be a monomial. The converse holds as well (by the same token). $\square$

- **Proof of (1) $\supseteq$ (3)**:

<u>Note</u>: The set $\mathcal{T}(V(f))$ is closed in the Euclidean Top (finite union of closed polyhedra) so to prove (1) $\supseteq$ (3) we can restrict to $\omega = -\mathrm{val}(y)$ with $y \in V(f)$.

But $f(y) = \sum_{u \text{ finite}} c_u y^u = 0$ so $\mathrm{val}(f(y)) = +\infty$ implies that there must be a tie among the summands with minimal valuation. Thus, $\exists\, u, u' \in \mathrm{supp}(f)$ with $u \neq u'$ & $\mathrm{val}(c_u y^u) = \mathrm{val}(c_{u'} y^{u'}) \leq \mathrm{val}(c_{\tilde{u}} y^{\tilde{u}})$ $\forall \tilde{u} \in \mathrm{supp}(f)$.

$$\Longleftrightarrow \quad \mathrm{val}(c_u) + \langle u, -\omega\rangle = \mathrm{val}(c_{u'}) + \langle u', -\omega\rangle \leq \mathrm{val}(c_{\tilde{u}}) + \langle \tilde{u}, -\omega\rangle \quad \forall \tilde{u}$$

$$\underset{\text{multiply by } (-1)}{\Longleftrightarrow} \quad W := -\mathrm{val}(c_u) + \langle u, \omega\rangle = -\mathrm{val}(c_{u'}) + \langle u', \omega\rangle \geq -\mathrm{val}(c_{\tilde{u}}) + \langle \tilde{u}, \omega\rangle \quad \forall \tilde{u}$$

so $\mathrm{trop}(f)_{(\omega)} = W$ & it's achieved at $u \neq u'$. So $\omega \in \mathcal{T}(V(f))$ $\square$

- **Proof of (3) $\subseteq$ (1) & $(-\mathrm{val}^{-1}(\omega))$ Zariski dense if f irreducible**: Follows from the next "lifting type" proposition plus the restriction of (1) to Γ_{val}^n.

Proposition 1: Fix $f \in K[x_1^{\pm}, \dots, x_n^{\pm}]$, $\overline{K} = K$, $\omega \in \Gamma^n_{val}$. Assume $in_\omega(f)$ is not a monomial & pick $\alpha \in (\mathbb{K}^*)^n$ with $in_\omega(f)(\alpha) = 0$. Then $\exists\, y \in (K^*)^n$ satisfying:

(i) $f(y) = 0$

(ii) $-val(y)_i = \omega_i \quad \forall i = 1, \dots, n$

(iii) $\overline{t^{\omega_i} y_i} = \alpha_i \quad \forall i = 1, \dots, n$

If f is irreducible, the set of such y's is dense in $V(f)$.

Proof: See the end of these notes.

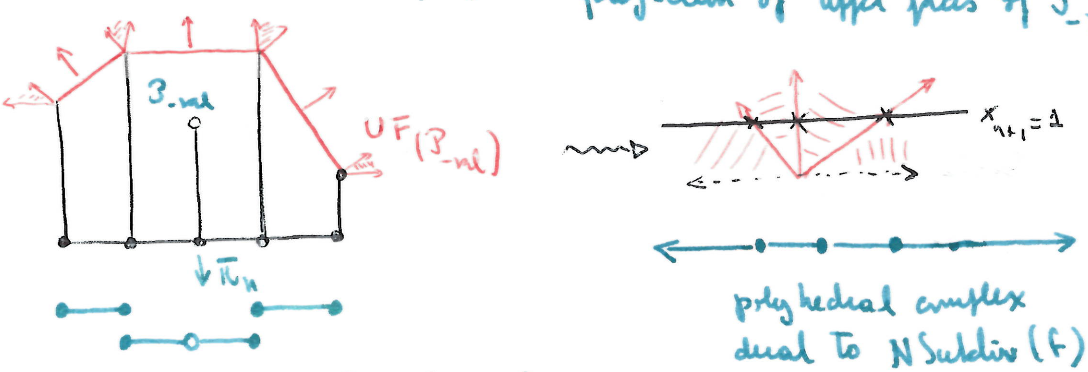
§2 Structure Theorem for Tropical hypersurfaces: duality with Newton subdivisions

Recall: Newton subdivision of f = reg. subdivision of $NP(f)$ induced by $\varphi(u) = -val(c_u)$

$P = NP(f) \longleftrightarrow \widetilde{\mathcal{P}} \subseteq \mathcal{P}_{-val} \subseteq \mathbb{R}^{n+1}$ $\qquad \mathcal{P}_{-val} = \operatorname{conv} \widetilde{\mathcal{P}} = \{(u, y) : u \in P \cap \mathbb{Z}^n,\ y \leq -val(c_u),\ c_u \neq 0\}$

$u \longmapsto (u, -val(c_u))$

- $UF(\mathcal{P}_{-val})$ = upper faces of $\mathcal{P}_{-val}$ =: $\{F \leq \mathcal{P}_{-val}$ induced by vectors $(v, 1)\}$.
- cells in $NSubdiv(f) \longleftrightarrow$ projection of upper faces of $\mathcal{P}_{-val}$.



$f = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$.

- Given $F \leq UF(\mathcal{P}_{-val})$: $\mathcal{N}(F) \cap \{x_{n+1} = 1\} = \{(v, 1)\ face_{(v,1)}(\mathcal{P}_{-val}) = F\}$ is a polyhedron.
- **Def**: $\widetilde{\pi}(\mathcal{N}(F)) = \{\omega \in \mathbb{R}^n : (\omega, 1) \in \mathcal{N}(F) \cap \{x_{n+1} = 1\}\} \subseteq \mathbb{R}^n$ is a polyhedron

Lemma 1: $\{\widetilde{\pi}(\mathcal{N}(F))\}_{F \leq UF(\mathcal{P}_{-val})}$ is a polyhedral complex (dual to $NSubd(f)$) of dimension n.

Proof: (1) $G, F \leq UF(\mathcal{P}_{-val})$ $\widetilde{\pi}(\mathcal{N}(F)) \cap \widetilde{\pi}(\mathcal{N}(G)) = \widetilde{\pi}(\mathcal{N}(\langle F, G \rangle))$ is a face of each because $\mathcal{N}(F) \geq \mathcal{N}(\langle F, G \rangle)$ & $\mathcal{N}(G) \geq \mathcal{N}(\langle F, G \rangle)$ ↳ minimal face containing F & G.

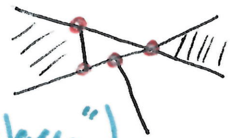
(2) $G \leq \widetilde{\pi}(\mathcal{N}(F))$ Need to show $G = \widetilde{\pi}(\mathcal{N}(F'))$ for $F' \geq F$.

This follows from the fact that $\mathcal{N}(\mathcal{P}_{-val})$ is a fan. & $\cap \{x_{n+1} = 1\}$ behaves well.

<u>Def</u>: Given a polyhedral complex Σ & $k \in \mathbb{N}$, we define the <u>k-skeleton</u> of Σ as

$$\Sigma^{(k)} = \{ \sigma \in \Sigma : \dim(\sigma) \leq k \}$$

Eg $k=0$: $\Sigma^{(0)}$ = vertices in Σ



Σ

<u>Prop 2</u> ("Structure Theorem for Tropical Hypersurfaces")

$\mathcal{T}(V(f))$ is the support of a <u>pure</u> Γ_{val}-rational polyhedral complex of dim $= n-1$ in $\mathbb{R}^n$. It is the $(n-1)$-skeleton of the dual complex to the Newton subdivision of f.
The multiplicities of the maximal cells = lattice length of the dual edge.
= # lattice pts in the edge -1.
The complex is balanced along the codimension-1 cells.

Here: a polyhedron σ is <u>Γ-rational</u> for $\Gamma \leq (\mathbb{R},+)$ subgroup if $\sigma = \{ \underline{x} : A\underline{x} \leq \underline{b} \}$, $A \in \mathbb{Q}^{m \times n}$, $b \in \Gamma^m$.

<u>Proof</u>: • By definition of $UF(\mathcal{P}_{-val}) \supseteq F$, Vertices of F are of the form $(u, -val(c_u))$ for $u \in supp(f)$.

• π_n (Vertices of F) = Vertices of $\pi_n(F)$.

$$F = face_{(v,1)}(\mathcal{P}_{-val}) = \{ x \in \mathcal{P}_{-val} : (v,1)\cdot x \geq (v,1)\cdot y \quad \forall y \in \mathcal{P}_{-val} \}$$

In particular, F not a vertex $\iff$ contains 2 vertices $(u, -val(c_u)), (u', -val(c_{u'}))$ & for any other $\tilde{u}$ in $supp(f)$:

$$(v,1)\cdot(u, -val(c_u)) = (v,1)\cdot(u', -val(c_{u'})) \geq (v,1)(\tilde{u}, -val(c_{\tilde{u}}))$$

$$\iff \quad v\cdot u - val(c_u) = v\cdot u' - val(c_{u'}) \geq v\tilde{u} - val(c_{\tilde{u}}) \iff v \in \mathcal{T}(V(f))$$

<u>Conclusion</u>: F is not a vertex $\iff \forall v \in \tilde{\pi}(\mathcal{N}(F)) : v \in \mathcal{T}(V(f))$.

• So $\mathcal{T}(V(f))$ supports the desired skeleton. (*)

• By construction, for any v in $\mathcal{T}(V(f))$ & $F = face_{(v,1)}(\mathcal{P}_{-val})$, $\tilde{\pi}(\mathcal{N}(F))$ is an Γ_{val}-polyhedron because it's defined by $\{ Ax \leq b \}$

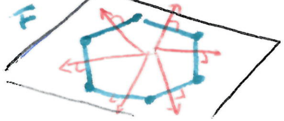
• A is determined by $\{ \tilde{u}-u, \tilde{u}-u' \}_{\tilde{u} \in supp(f)}$. so $A \in \mathbb{Q}^{m \times n}$

• b " " " $\{ val(c_u) - val(c_{\tilde{u}}) = val(c_u / c_{\tilde{u}})$, $val(c_{u'}) - val(c_{\tilde{u}}) = val(c_{u'}/c_{\tilde{u}}) \}$ so $b \in \Gamma_{val}^m$.

(*) • $\tilde{\pi}_n(\mathcal{N}(F))$ has dim $= n \iff F$ is a vertex of $UF(\mathcal{P}_{-val})$.

<u>Balancing</u>: codim-1 cell in $\mathcal{T}(V(f))$ is dual to a bounded 2-face F in $UF(\mathcal{P}_{-val})$

F



balancing $= \sum_{\sigma \supseteq F} \vec{n}_{\sigma|F} m_{Trop}(\sigma) = \vec{0}$ $\qquad \vec{n}_{\sigma|F}$ = primitive normal to edge dual to σ.
following because the sum of the edges $= \vec{0}$.

Proof of "lifting" Proposition (Prop 2)

We proceed in two steps.

Useful properties (1) $\operatorname{trop}(fg)_{(w)} = \operatorname{trop}(f)_{(w)} + \operatorname{trop}(g)_{(w)}$ $\forall w \in \mathbb{R}^n$

(2) $\operatorname{in}_w(fg) = \operatorname{in}_w(f)\operatorname{in}_w(g)$

[For a proof of these 2 statements, see Lemmas 0 & 1 in Lecture VII.]

- **Base case**: $n=1$. We write in Lemma 2.
- **General n**: It is proved by reduction to the case where no two monomials in f are divisible by the same (max'l) power of x_n: ie x^u & x^v in $\operatorname{supp}(f)$, then $u_n \neq v_n$.

The reduction argument is the content of Lemma 4.

The proof of the inductive step under this assumption is the content of Lemma 3.

- The claim of density when f is irreducible uses the (~~special~~) density statement when $f=0$. This is the content of Lemma 5. The proof will use a projection to a lower dimensional torus under the reduction argument, as we now show:

Write
$$\boxed{f = \sum_{k\in\mathbb{Z}_{\geq 0}} a_k \underline{x}'^{\alpha_k} x_n^k \quad \text{for } a_k \in K \quad \alpha_k \in \mathbb{Z}^{n-1}_{\geq 0} \quad \underline{x}' = (x_1, \ldots, x_{n-1})} \qquad (*)$$

Set $\mathcal{Y} = \{\underline{y} \in (K^\times)^n : -\operatorname{val}(\underline{y}) = \underline{w} \ \& \ \overline{t^{w_i} y_i} = \alpha_i \ \forall i = 1, \ldots, n\} \cap V(f)$

Want to show f irreducible $\Rightarrow$ $\mathcal{Y}$ is Zariski dense in $V(f)$.

We proceed by reduction to the $n=1$ case.

- **Base case** $n=1$. Since $\overline{K} = K$, $f = ax+b$ & $\operatorname{in}_w f$ has a root in $\Bbbk^\times$, we conclude $b \neq 0$ & $\operatorname{in}_w f = \overline{t^{-\operatorname{val} a} a}\, x + \overline{t^{-\operatorname{val}(b)} b}$ is not a monomial & $\alpha = \dfrac{\overline{t^{-\operatorname{val}(b)} b}}{\overline{t^{-\operatorname{val}(a)} a}}$.
  - $y = -b/a$ is the unique root of f & $y \neq b$.
  - $-\operatorname{val}(a) + w = -\operatorname{val}(b)$ so $w = \operatorname{val}(a) - \operatorname{val}(b) = \operatorname{val}(a/b) = -(\operatorname{val}(-b/a))$
  - $t^{-w} y = \dfrac{t^{-\operatorname{val}(b)} b}{t^{-\operatorname{val}(a)} a}$ so $\overline{t^{-w} y} = \dfrac{\overline{t^{-\operatorname{val}(b)} b}}{\overline{t^{-\operatorname{val}(a)} a}} = \alpha$

so $\mathcal{Y} = V(f)$ & so density follows.

- **General n (ind step)**: By Lemma 5, the set $\mathcal{Y}' = \{y \in (K^\times)^{n-1} : -\operatorname{val}(y_i) = w_i,\ \overline{t^{-w_i} y_i} = \alpha_i$ for $i = 1, \ldots, n-1\}$ is Zariski dense in $(K^\times)^{n-1}$. We consider the projection to the first $(n-1)$ coordinates

$$\begin{tikzcd}
\pi_{n-1} : (K^\times)^{n} \arrow[r] & (K^\times)^{n-1} \\
\mathcal{Y} \arrow[u, phantom, "\cup" rotate=90] \arrow[r] \arrow[ur, phantom, "\circlearrowleft"] & \mathcal{Y}' \arrow[u, phantom, "\cup" rotate=90]
\end{tikzcd}
\qquad
\begin{array}{l} y' = (y_1, \ldots, y_{n-1}) \\ y = (y_1, \ldots, y_{n-1}, y_n) \end{array}$$

& the Proof of Lemma 4

From (*), we can lift any point y' in $\mathcal{Y}'$ to a point y in $V(f) \cap (K^\times)^{n-1}$ with $-\operatorname{val}(y_n) = w_n$, $\overline{t^{w_n} y_n} = \alpha_n$

Since $\mathcal{Y}'$ is Z. dense in $(K^*)^{n-1}$, $\pi_{n-1}(\mathcal{Y})$ is not contained in any hypersurface $V(h)$ in $(K^*)^{n-1}$. We claim that $\mathcal{Y} \xrightarrow{\pi_n} \mathcal{Y}'$ must be Zariski dense in $V(f)$.

If not, $\exists\, g \in K[x_1^{\pm}, \ldots, x_n^{\pm}]$ with $\mathcal{Y} \subseteq V(g)$

So $\langle f, g\rangle \cap K[x_1^{\pm}, \ldots, x_{n-1}^{\pm}] = I$ satisfies $(K^*)^{n-1} = \overline{\mathcal{Y}'} \subseteq V(I)$, so $I = 0$

Thus, if $\mathbb{F} = \operatorname{Quot}(K[x_1^{\pm}, \ldots, x_n^{\pm}])$, we conclude that $\langle f, g\rangle \mathbb{F}[x_n^{\pm}] \neq \langle 1\rangle$

Furthermore f irreducible over $K[x_1^{\pm}, \ldots, x_n^{\pm}] \Rightarrow f$ is irred over $\mathbb{F}[x_n^{\pm}]$.

So $\langle f, g\rangle \mathbb{F}[x_n^{\pm}] = \langle f\rangle$ & so g is a multiple of f over $\mathbb{F}[x_n^{\pm}]$.

By using the UFD property of $K[x_1^{\pm}, \ldots, x_n^{\pm}]$ & clearing denominators from $\mathbb{F}$ if needed, we conclude that $f \mid g$ over $K[x_1^{\pm}, \ldots, x_n^{\pm}]$.

We conclude $\mathcal{Y} \subseteq V(f)$ is Z. dense. $\square$.

Lemma 2: Statements (i) — (iii) hold for $n=1$.

Proof: After multiplying by some x^a we may assume $f = \sum_{k=0}^{s} a_k x^k \quad a_0, a_s \neq 0$

We factor f in $K[x]$: $f = c \prod_{j=1}^{s} (x - \beta_j)$ with all $\beta_j \neq 0$.

By the mult. property of in_ω, we get $\mathrm{in}_\omega(f) = t^{-\mathrm{val}(c)} \bar{c} \prod_{j=1}^{s} \mathrm{in}_\omega(x - \beta_j)$

since α is a root of $\mathrm{in}_\omega(f)$, we know some j has $\mathrm{in}_\omega(x - \beta_j) = x - \overline{t^{-\mathrm{val}(\beta_j)} \beta_j}$.

$\neq 0$ (ie, it is not a monomial) & $\mathrm{in}_\omega(x-\beta_j)$ vanishes at α, so $\omega = -\mathrm{val}(\beta_j)$ & $y = \beta_j$ satisfies $\overline{t^{+\omega} y} = \alpha$ & $-\mathrm{val}(y) = \omega$ & $f(y) = 0$ $\square$

Lemma 3 Assume f has the expression (*) Then (i) — (iii) holds for f.

Proof: We consider the set

$$\mathcal{Y}' = \{ y' \in (K^*)^{n-1} : -\mathrm{val}(y_i) = \omega_i,\ \overline{t^{-\omega_i} y_i} = \alpha_i \text{ for } i = 1, \ldots, n-1\}$$

By Lemma 5, this set is dense in $(K^*)^{n-1}$. The special form of (*) ensures that for any $y' \in \mathcal{Y}'$ $g_{(x_n)} := f(y', x_n) \in K[x_n]$ is never the zero polynomial & it's also not a monomial

We consider the projection $\pi_n : \mathbb{Z}^n \longrightarrow \mathbb{Z}^{n-1}$, $u \longmapsto u'$ if $u = (u', u_n)$, $u' = (u_1, \ldots, u_{n-1})$

Write $g = \sum_i d_i x_n^i$ & $d_i = c_u y'^{u'}$ for a unique u in $\mathrm{supp}(f)$, "(u', i)"

(again, this is ensured by the special form (*) for f)

<u>Claim</u>: $\mathrm{trop}(g)(w_n) = \mathrm{trop}(f)(w)$.

Pf/ $\mathrm{trop}(g)(w) = \max_i \{-\mathrm{val}(d_i) + i w_n\} = \max_i \{-\mathrm{val}(c_u) - \mathrm{val}(y'^{u'}) + i w_n\}$

$$= \max_i \{-\mathrm{val}(c_u) + \langle u', \underbrace{-\mathrm{val}(y')}_{= w'} \rangle + i w_n\} = -\max_u \{-\mathrm{val}(c_u) + \langle w, u \rangle\}$$

$= \mathrm{trop}(f)(w)$.

$u = (u', i)$

$w = (w', w_n)$

• From this proof, we observe that the u's realizing $\mathrm{trop}(f)_{(w)}$ are exactly the ones of the form (u', i) where i realizes $\mathrm{trop}(g)(w_n)$. Call the former Λ. & $\pi: \mathbb{Z}^n \to \mathbb{Z}$, $(u', i) \mapsto i$

Then
$$\mathrm{in}_{w_n}(g) = \sum_{i \in \pi(\Lambda)} \overline{t^{-\mathrm{val}(d_i)} d_i}\, x_n^i$$
$$= \sum_{i \in \pi(\Lambda)} \overline{t^{-\mathrm{val}(c_{(u',i)})} c_{(u',i)} t^{-\langle u', w' \rangle} y'^{u'}}\, x_n^i$$
$$= \sum_{u \in \Lambda} \overline{t^{-\mathrm{val}(c_u)} c_u}\, \alpha'^{u'} x_n^i \quad \text{for } \alpha' = (\alpha_1, \ldots, \alpha_{n-1}).$$

$$\left[\text{because } t^{-\langle u', w' \rangle} y'^{u'} = \prod_{j=1}^{n-1} (\underbrace{t^{-w_j} y_j}_{\mathrm{val}=0})^{u'_j} \Rightarrow \overline{t^{-\langle u', w' \rangle} y'^{u'}} = \prod_{j=1}^{n-1} \overline{\alpha_j}^{u'_j} = \alpha'^{u'}\right]$$

$$= \sum_{u \in \Lambda} \overline{t^{-\mathrm{val}(c_u)} c_u}\, x^u \Big|_{(x_1, \ldots, x_{n-1}) = \alpha'} = \mathrm{in}_w f\, (\alpha', x_n)$$

We conclude $\mathrm{in}_{w_n}(g)(\alpha_n) = \mathrm{in}_w f(\alpha) = 0$.

By Lemma 3 applied to g, we can find $y_n \in K^*$ with $-\mathrm{val}(y_n) = w_n$, $g(y_n) = 0$ & $\overline{t^{-w_n} y_n} = \alpha_n$.

Therefore $y = (\underbrace{y_1, \ldots, y_{n-1}}_{= y'}, y_n)$ satisfies $f(y) = g(y_n) = 0$, $\overline{t^{-w} y} = \alpha$, $\mathrm{val}(y) = -w$. ☐

<u>Lemma 4</u>: Statements (i) — (iii) hold for general f, if they hold in the special case of $(*)$.

<u>Proof</u>: Given any $f \in K[x_1^{\pm}, \ldots, x_n^{\pm}]$ (not a monomial), we apply an automorphism φ_ℓ of $(K^*)^n$ (monomial map with exponents in $\mathbb{Z}^n$) to put f in the desired form,

$\tilde{f}(x) := f(\varphi_\ell^*(x))$ where $\varphi_\ell^*: K[x_1^{\pm}, \ldots, x_n^{\pm}]$ is defined by $\varphi_\ell^*(x_j) = \begin{cases} x_j x_n^{\ell^j} & j < n \\ x_n & j = n \end{cases}$

For $u' \in \mathbb{Z}^{n-1}$ we have $\varphi_\ell^*(x'^{u'} x_n^i) = x'^{u'} x_n^{(i + \sum u_j \ell^j)}$

For $l >> 0$, all the exponents $(i+\sum_{j=1}^{n-1} u_j l^j)$ are expressions in base l of different elements in $\mathbb{Z}$, so $\tilde F_{(x)}$ has property (*).

We write $(\varphi_l^*)^{-1}(x_j) = \begin{cases} x_j\, x_n^{-l^j} & \text{for } j<n \\ x_n & j=n \end{cases}$, $\boxed{\tilde\alpha = (\varphi_l^*)^{-1}(\alpha)}$

We set $\boxed{\tilde w \in \mathbb{R}^n}$ as $\tilde w_j = \begin{cases} w_j - l^j w_n & \text{for } j<n \\ w_n \end{cases}$ (think of applying valuation to $\varphi_l^{*-1}(y)$.)

Each term $c_{u,i}\, x^{(u,i)}$ in f becomes $c_{u,i}\, x'^{u'} x_n^{i+\sum u_j l^j}$, so

$$-val(c_{u,i}) + \langle w,(u,i)\rangle \quad \text{becomes} \quad -val(c_{u,i}) + \langle \tilde w', u'\rangle + \tilde w_n\Big(i+\sum_{j=1}^{n} u_j l^j\Big)$$
$$= -val(c_{u,i}) + \Big\langle \tilde w, \big(u', i+\sum_{j=1}^{n-1} u_j l^j\big)\Big\rangle$$
$$= -val(c_{u,i}) + \langle w,(u,i)\rangle$$

so $in_{\tilde w}(\tilde F)_{(x)} = (in_w(f))(\varphi_l^*(x))$.

- Thus if α is a root of $in_{\tilde w}(f)$ in $(K^*)^n$, then $\tilde\alpha$ is a root of $in_{\tilde w}\tilde F$ in $(K^*)^n$.
- y is a root of f in $(K^*)^n \iff \boxed{\tilde y = \varphi_l^{*-1}(y)}$ is a root of $\tilde F$ in $(K^*)^n$

Note:

$-\tilde w_j = val(\tilde y_j) = val(y_j) - l^j\, val(y_n) = val(y_j) - l^j(-w_n)$ $\forall j<n$ so $val(y_j) = -\tilde w_j - l^j \tilde w_n = -w_j$ ✓

$-\tilde w_n = val(\tilde y_n) = val(y_n)$ so $val(y_n) = -w_n$ ($-\tilde w_n = -w_n$)

$$val(y) = -w \iff val(\tilde y) = -\tilde w.$$

Furthermore (1) $t^{+\tilde w_j}\tilde y_j = t^{w_j - l^j w_n} y_j\, y_n^{-l^j} = t^{w_j} y_j \big(\underbrace{t^{w_n} y_n}_{\in \mathcal{O}_v^*}\big)^{-l^j}$ $j<n$

so $\tilde\alpha_j = \overline{t^{\tilde w_j}\tilde y_j} = \overline{t^{w_j} y_j}\,\big(\overline{t^{w_n} y_n}\big)^{-l^j} = \overline{t^{w_j} y_j}\,\alpha_n^{-l^j}$ for $j<n$ (**)

(2) $t^{+\tilde w_n}\tilde y_n = t^{w_n} y_n$ so $\tilde\alpha_n = \overline{t^{\tilde w_n}\tilde y_n} = \overline{t^{w_n} y_n} = \alpha_n$ ✓

From (**) we get $\overline{t^{w_j} y_j} = \alpha_n^{l^j}\tilde\alpha_j = \tilde\alpha_n^{l^j}\tilde\alpha_j = \alpha_j$ (for $j<n$) ✓ & so $\alpha_j = \overline{t^{w_j} y_j}$ $\forall j$.

We conclude, if the statements (i)–(iii) hold (for $\tilde\alpha, \tilde w$) for $\tilde F$, they hold for f (for α, w) $\square$.

Lemma 5: Given n & $\alpha \in (\mathbb{K}^*)^n$ & $w \in \Gamma_{val}^n$, the set

$$\mathcal{Y}_{\alpha,w} = \{ y \in (\mathbb{K}^*)^n : -val(y_j) = w_j, \quad \overline{t^{+w_j} y_j} = \alpha_j \quad \forall j = 1, \dots, n \}$$

is Zariski dense in $(\mathbb{K}^*)^n$.

Proof: To prove the result, we must show that for any $h \in \mathbb{K}[x_1^{\pm}, \dots, x_n^{\pm}]$ we can find $y \in \mathcal{Y} =: \mathcal{Y}_{\alpha,w}$ with $h(y) \neq 0$. If h is a monomial, there's nothing to show. We proceed by induction on n; assume h is not a monomial.

• Base case: $n=1$. Given $\alpha \in \mathbb{K}^*$, we have infinitely many $z \in \mathcal{O}_v^*$ with $\overline{z} = \alpha$, namely $w \in \alpha + \mathfrak{m}$ & we can pick $\alpha + at^i$, $i \in \Gamma_{val} \setminus \{0\}$, $a \in \mathbb{K}^*$.

For any h, we pick $t^{-w} z \notin V(h)$ with $\overline{z} = \alpha$ (the later is an infinite set & $V(h)$ is finite). ✓

• Inductive Step: Write $h = x_n^a \sum_{j=0}^{s} h_j x_n^j$ for $h_j \in \mathbb{K}[x_1^{\pm}, \dots, x_{n-1}^{\pm}]$, $a \in \mathbb{Z}$, $s > 0$.

By the Inductive hypothesis, we have $y' = (y_1, \dots, y_{n-1}) \in (\mathbb{K}^*)^{n-1}$ with $val(y_i') = -w_i \quad \forall i = 1, \dots, n-1$, $\overline{t^{w_i} y_i} = \alpha_i \quad \forall i = 1, \dots, n-1$ and

$h_j(y') \neq 0 \quad \forall j = 0, \dots, s$. ($\mathcal{Y}_{\alpha', w'}$ is Z. dense in $(\mathbb{K}^*)^{n-1}$)

Then $\tilde{h}(x_n) = x_n^a \sum_{j=0}^{s} \underset{\neq 0 \ \forall j}{h_j(y')} x_n^j$ is an univariate polynomial. By the $n=1$ case, we can find y_n with $-val(y_n) = w_n$, $\overline{t^{w_n} y_n} = \alpha_n$ & $\tilde{h}(y_n) \neq 0$.

Then $y = (y', y_n)$ satisfies $y \in \mathcal{Y}_{\alpha,w}$ & $h(y) \neq 0$, as we wanted.