Lecture VII : Gröbner basis over valued fields

Recall: $0 \neq f = \sum_{u \text{ finite}} c_u x^u \in K[x_1^{\pm}, \dots, x_n^{\pm}] = S$ (K, val) valued field with a splitting $\Gamma_{val} \ni \gamma \longmapsto t^{\gamma} \in K^{\times}$ (homomorphism).

We define $$in_\omega(f) = \sum_u \overline{t^{trop(f)(\omega) - \langle \omega, u\rangle} c_u}\, x^u \in \Bbbk[x_1^{\pm}, \dots, x_n^{\pm}] \text{ for } \omega \in \Gamma_{val}^n$$

$$\stackrel{\text{Lecture V}}{=} \sum_{u \in \Lambda} \overline{t^{-val(c_u)} c_u}\, x^u \qquad \text{where } \Lambda = \{u \mid trop(f)(\omega) = -val(c_u) + \langle \omega, u\rangle\}$$
for $\omega \in \mathbb{R}^n$.

Note: In the first definition, we need $\omega \in \Gamma_{val}^n$ because otherwise there is no garantee that $trop(f)(\omega) - \langle \omega, u\rangle \in \Gamma_{val}$, so can't use the splitting ($= -val(c_u)$ for $u \in \Lambda_\omega$ so only for those exponents u, $t^{\square}$ is well-defined!)

Remark: $$\begin{aligned} K^{\times} &\longrightarrow \Bbbk^{\times} \\ a &\longmapsto \overline{t^{-val(a)} a} \end{aligned}$$ is a homomorphism of multiplicative ab gps

Lemma 0: Given $f, g \in S \setminus \{0\}$ $\quad trop(fg) = trop(f) + trop(g)$

Proof: Write $g = \sum_v d_v x^v \rightsquigarrow fg = \sum_a \left(\sum_{u+v=a} c_u d_v\right) x^a$

$$trop(fg)(\omega) = \max_a \Big(\underbrace{-val\Big(\sum_{u+v=a} c_u d_v\Big)}_{= -\infty \text{ if } a \notin supp(fg)} + \langle \omega, a\rangle \Big)$$

but $fg \neq 0$, so at least one term in $supp(fg)$

$$\stackrel{\substack{\text{Claim}\\ \downarrow}}{=} \max_u \left(-val(c_u) + \langle u, \omega\rangle\right) + \max_v \left(-val(d_v) + \langle v, \omega\rangle\right)$$
$$= trop(f)(\omega) + trop(g)(\omega).$$

Pf of Claim:

$(\leq)$ $$val\Big(\sum_{u+v=a} c_u d_v\Big) \geq \min_{u+v=a} \{val(c_u) + val(d_v)\}$$

$(*)$ $$\langle \omega, a\rangle - val\Big(\sum_{u+v=a} c_u d_v\Big) \leq \langle \omega, a \rangle - \min_{u+v=a}\{val(c_u) + val(d_v)\}.$$

$$\underset{\substack{\downarrow \\ \langle \omega, a\rangle = \langle u, a\rangle + \langle v, a\rangle}}{=} -\{\min_{u+v=a} val(c_u) + val(d_v)\} + \langle \omega, u\rangle + \langle \omega, v\rangle \underset{\substack{\downarrow \\ -\min = \max -(\,)}}{=} \max_{u+v=a}\{-val(c_u) - val(d_v)\} + \langle \omega, u\rangle + \langle \omega, v\rangle$$

$$= \max_{u+v=a} \{(-val(c_u) + \langle \omega, u\rangle) + (-val(d_v) + \langle \omega, v\rangle)\}$$

$\leq \max_{u,v} \{ -val(c_u) + \langle w, u\rangle + (-val(d_v) + \langle w, v\rangle \} = \max_u + \max_v$

$= trop(f)(w) + trop(g)(w)$

So now take max in a in (*) & get $trop(fg)(w) \leq trop(f)(w) + trop(g)(w)$

($\geq$) If $w \notin \mathcal{T}(V(f)) \cup \mathcal{T}(V(g))$, then $trop(f)(w)$ & $trop(g)(w)$ are achieved at unique monomials $c_u x^u$, $d_v x^v$ for $u \in supp(f)$, $v \in supp(g)$

We claim: $c_u d_v x^{u+v}$ is a term in fg & so $trop(fg)(w) \geq trop(c_u d_v x^{u+v}) = trop(f)(w) + trop(g)(w)$

For this, it's enough to show that this term won't cancel with any others: Take $u+v=a$ & $\left(\sum_{\tilde u + \tilde v = a} c_{\tilde u} d_{\tilde v}\right)$ = coeff of x^a in fg.

If coeff $=0$, then $\exists\, \tilde u \neq \tilde u'$ in $supp(f)$ & $\tilde v \neq \tilde v'$ in $supp(g)$ with $(\tilde u, \tilde v) \neq (u,v)$

and $-val(c_{\tilde u}) - val(d_{\tilde v}) = -val(c_{\tilde u'}) - val(d_{\tilde v'}) \geq -val(c_{\tilde{\tilde u}}) - val(d_{\tilde{\tilde v}})$ (**)

(because the min valuations of the summands have to cancel!) $\underline{\forall}\ \tilde{\tilde u} + \tilde{\tilde v} = a$

since $(\tilde u, \tilde v) \neq (u,v)$

But $-val(c_{\tilde u}) - val(d_{\tilde v}) + \langle a, w\rangle = (-val(c_{\tilde u}) + \langle \tilde u, w\rangle) + (-val(d_{\tilde v}) + \langle \tilde v, w\rangle)$

& $< -val(c_u) + \langle u, w\rangle + (-val(d_v) + \langle v, w\rangle)$ & $u+v=a$

This contradicts (**)! □

But the complement of $\mathcal{T}(V(f)) \cup \mathcal{T}(V(g))$ is a union of n-dim'l polyhedra in $\mathbb{R}^n$ (see Lecture VI) and $trop(fg)(w) = trop(f)(w) + trop(g)(w)$ there.

Since both sides are continuous functions in $\mathbb{R}^n$, the equality holds in $\mathbb{R}^n$ □

Lemma 1: Given $f, g \in S \setminus \{0\}$, $in_w(fg) = in_w(f)\, in_w(g)$ $\forall w \in \mathbb{R}^n$.

Proof: By Lemma 0, $trop(fg)(w) = trop(f)(w) + trop(g)(w)$ and

$I_{f,w} + I_{g,w} = \{ u+v \mid trop(f)(w) = -val(c_u) + \langle u, w\rangle \ \&\ trop(g)(w) = -val(d_v) + \langle w, v\rangle \}$

(*) $= \{ a \in supp(fg) \mid trop(fg)(w) = -val(coeff_{x^a}(fg)) + \langle w, a\rangle \} =: I_{fg,w}$

so $in_w(fg) = \sum_{a \in I_{fg,w}} \overline{t^{-val(coeff_{x^a}(fg))}\, coeff_{x^a}(fg)}\; x^a$

& $coeff_{x^a}(fg) = \sum_{u+v=a} c_u d_v$ & by (*) $-val(coeff_{x^a}(fg)) = -val(c_u) - val(d_v)$ for $u+v=a$, $u \in I_{f,w}$, $v \in I_{g,w}$

$\Rightarrow in_w(fg) = \sum_{a \in I_{fg,w}} \left(\sum_{\substack{u+v=a \\ u \in I_{f,w} \\ v \in I_{g,w}}} \overline{c_u t^{-val(c_u)}\, d_v t^{-val(d_v)}} \right) x^a$

↓ defn + (*)

$$= \sum_{a \in I_{fg,w}} \sum_{\substack{u \in I_{f,w} \\ v \in I_{g,w} \\ u+v=a}} \overline{c_u t^{-\mathrm{val}(c_u)}}\ \overline{d_v t^{-\mathrm{val}(d_v)}}\ x^{a}$$

Remark[p13]

$$\overset{\text{by } (*)}{=} \sum_{u \in I_{f,w}} \overline{c_u t^{-\mathrm{val}(c_u)}}\, x^u \sum_{v \in I_{g,w}} \overline{d_v t^{-\mathrm{val}(d_v)}}\, x^v$$

$$= \mathrm{in}_w(f) \cdot \mathrm{in}_w(g)$$

and the result holds □.

Note: If K has trivial valuation, then the proof is much simpler since the twistings don't play a role (in other words, we will have $\overline{\text{coeff}} = 0$ if and only if $\text{coeff} = 0$)

- From now on, we shift from Laurent polynomials to honest polynomials & restrict to homogeneous ideals I $\subseteq R = K[x_0, \ldots, x_n]$

Def: $I \subseteq R$ is homogeneous if $\forall f = \sum_i f_{d_i}$ where f_{d_i} is homogeneous polynomial of $\deg f_i = d_i$, we have $f \in I \Longrightarrow f_{d_i} \in I \ \forall i$.

Note: Here, homogeneous refers to the standard grading, ie $\deg(x_i) = 1$ $\forall i = 0, \ldots, n$

So $\deg(x^\alpha) = |\alpha| := \alpha_0 + \cdots + \alpha_n$. $\forall \alpha \in \mathbb{N}_0^{n+1}$.

Later in the course, we'll consider other gradings, including multigradings.

Def: Multigrading induced by a matrix $A \in \mathbb{Z}^{m \times (n+1)}$

$$\deg_A(x^\alpha) = A\alpha \in \mathbb{Z}^m$$

We will relate these multigrading & homogeneous ideals wrt these multigradings to torus actions of $(K^\times)^m$ induced by the lattice of rows of A.

Here, Multihomogeneous = A-homogeneous

Eg: $f = x^3 - yz^2$ is A-homog for $A = \begin{bmatrix} 1 & 3 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ & homogeneous degree $= (3,3)$.

Def: If $I \subseteq K[x_0, \ldots, x_n]$ is homogeneous, we define its <u>initial ideal</u> for $w \in \mathbb{R}^n$

$$in_w(I) = \langle in_w(f) : f \in I \rangle \subseteq K[x_1, \ldots, x_n]$$

Def A set $G = \{g_1, \ldots, g_s\} \subset I$ is a <u>Gröbner basis for I</u> with respect to w if $in_w(I) = \langle in_w(g_1), \ldots, in_w(g_s) \rangle$

<u>Remark</u>: Traditional Gröbner basis theory uses <u>term orders</u> (or <u>monomial orders</u>)
- total order
- $1 < x^\alpha \quad \forall \alpha \Leftrightarrow$ well-order
- $in_<(f)$ is always a monomial
- $x^\alpha < x^\beta \Rightarrow x^{\gamma+\alpha} < x^{\gamma+\beta}$

& works with trivial valuation.

<u>Examples</u>
- $\leq_{lex}$ $\quad x^\alpha \leq_{lex} x^\beta$ if first nonzero entry in $\beta - \alpha$ is > 0.
- $\leq_{dlex}$ $\quad x^\alpha \leq_{dlex} x^\beta$ if $\underset{(deg)}{|\alpha| < |\beta|}$ or $|\alpha| = |\beta|$ & $x^\alpha \leq_{lex} x^\beta$.
- $\leq_{revlex}$ $\quad x^\alpha \leq_{revlex} x^\beta$ if $\underset{(deg)}{|\alpha| < |\beta|}$ or $|\alpha| = |\beta|$ & the <u>last</u> nonzero entry in $\beta - \alpha$ is <u>negative</u>

<u>Eg.</u> $x_1 x_3 >_{dlex} x_2^2$ but $x_1 x_3 <_{revlex} x_2^2$

extends our earlier notion →
- weight orders $\leq_w$: $x^\alpha \leq_w x^\beta$ if $\langle w, \alpha \rangle \leq \langle w, \beta \rangle$

  w <u>generic</u> means we want $in_w(f)$ to always be a single monomial.
- block orders = Take orders on a partition of the variables & use lex order for the blocks.

<u>Prop</u> (Robbiano, [1986]) Every monomial order in R is a lexicographic product of at most $n+1$ weight orders $\leq_{w_i}$.

<u>Prop</u> [Bayer [1982]] Any monomial order $<$ can be approximated by a single weight order $\leq_w$; ie given a finite set of pairs of monomials $(m_i < n_i)$, we can find w such that $m_i \leq_w n_i$ & $\leq_w$ is compatible with $<$ (ie. $m_i <_w n_i \Rightarrow m_i < n_i$)

<u>Why?</u> Can use these monomial orders to develop a <u>division algorithm</u> on R. Using this:
- gives an algorithm to compute Gröbner basis wrt. $<$ (<u>Buchberger's algorithm</u>)
- Compute Syzygies, regular sequences on R-modules.
- Construct flat degenerations
- Test ideal membership
- Compute Hilbert functions
- Do elimination, given $\{x_{i_1}, \ldots, x_{i_s}\}$ compute $I \cap K[x_{i_1}, \ldots, x_{i_s}]$.
- Do saturation, projective closures, compute radicals, ...

[See <u>Eisenbud's Commutative Algebra book</u> [Chapter 15] for more on this...]