Lecture VIII: Gröbner basis over valued fields II

Last time: $f, g \in K[x_1^{\pm}, \ldots, x_n^{\pm}] \setminus \{0\}$, then $\operatorname{trop}(fg)_{(\omega)} = \operatorname{trop}(f)_{(\omega)} + \operatorname{trop}(g)_{(\omega)}$

Use this to show $\operatorname{in}_\omega(fg) = \operatorname{in}_\omega(f)\operatorname{in}_\omega(g)$ $\forall \omega \in \mathbb{R}^n$.

Q: What about the behavior of $\operatorname{trop}(\,)$ & in_ω with respect to addition?

Example: $n=1$ $f = 1+\underline{x}$, $g = 2-\underline{x}$ $\omega = (1) \in \mathbb{R}^1$. $K = \mathbb{C}$ with trivial valn.

- $\operatorname{trop}(f)_{(\omega)} = \max\{0, 1\} = 1$ & $\operatorname{trop}(g)_{(\omega)} = \max\{0,1\} = 1$, $\operatorname{Trop}(\overset{=3}{f+g})_{(\omega)} = 0$
- $\operatorname{in}_\omega(f) = \sum\limits_{\langle u,1\rangle = 1} c_u t^{-\operatorname{val}(c_u)} x^u = x$, $\operatorname{in}_\omega(g) = -x$, $\operatorname{in}_\omega(f+g) = \operatorname{in}_\omega(3) = 3$.

So $\operatorname{trop}(f+g)_{(\omega)} < \max\{\operatorname{trop}(f)_{(\omega)}, \operatorname{trop}(g)_{(\omega)}\}$ & $\operatorname{in}_\omega(f+g) \neq \operatorname{in}_\omega(f) + \operatorname{in}_\omega(g)$
(in general: $\leq$)

Lemma 1: Assume $\{f_\alpha\}_{\alpha\in A}$ be a finite collection in $K[x_1^{\pm}, \ldots, x_n^{\pm}]\setminus\{0\}$ with pairwise disjoint supports. Fix $f = \sum_{\alpha \in A} f_\alpha$, $W = \operatorname{trop}(f)_{(\omega)}$, $W_\alpha = \operatorname{trop}(f_\alpha)_{(\omega)}$. Then

$$\operatorname{in}_\omega(f) = \sum_{\substack{\alpha \\ W_\alpha = W}} \operatorname{in}_\omega(f_\alpha) \qquad (*)$$

Pf/ Since the supports are pairwise disjoint, no cancellations occur & we conclude that $\operatorname{trop}(f)_{(\omega)} = \max\limits_\alpha W_\alpha$. The identity $(*)$ follows by definition of ω-initial forms. □

- For this, it's convenient to restrict to homogeneous ideals ($f = \sum_i f_i$, f_i homogeneous of $\deg(f_i) = i$. in $K[x_0, \ldots, x_n]$ $f \in I \Longrightarrow f_i \in I$ $\forall i$)

Equiv: I has a set of homogeneous generators $\iff$ I homog.

Lemma 2: Fix $I \subset K[x_0, \ldots, x_n]$ homogeneous ideal, $\omega \in \mathbb{R}^{n+1}$. Then $\operatorname{in}_\omega I$ is a homogeneous ideal in $\Bbbk[x_0, \ldots, x_n]$ ($\Bbbk = \mathcal{O}_v / \mathfrak{m}_v$) & we can find a homogeneous Gröbner basis for I.

Proof: • Homogeneity: Write $I \ni f = \sum_i f_i$ f_i homog of deg i, so $\{f_i\}$ have pairwise disjoint supports. Then by Lemma 1, $\operatorname{in}_\omega(f) = \sum\limits_{\operatorname{trop}(f_i)_{(\omega)} = \operatorname{trop}(f)_{(\omega)}} \operatorname{in}_\omega(f_i)$. $\operatorname{in}_\omega(f_i)$ homog of deg i $\Rightarrow$ This is the decomp. of $\operatorname{in}_\omega(f)$ in homog pieces.

Since I homogeneous $f_i \in I$ $\forall i$, $\operatorname{in}_\omega(f_i) \in \operatorname{in}_\omega(I)$ $\forall i$.

In particular, $\operatorname{in}_\omega I = \langle \operatorname{in}_\omega(f) : f \in I \text{ homog} \rangle \subseteq \Bbbk[x_0, \ldots, x_n]$ is gen. by homog polys, so $\operatorname{in}_\omega I$ is homog.

- Homog Gröbner basis:

Recall: $\{g_1, \ldots, g_s\} \subseteq I$ is a GB if $\operatorname{in}_\omega(I) = \langle \operatorname{in}_\omega(g_1), \ldots, \operatorname{in}_\omega(g_s)\rangle$.

Since $\Bbbk[x_0, \ldots, x_n]$ is Noetherian, among the set $\{\operatorname{in}_\omega(f) : f \in I \text{ homog}\}$ which generates $\operatorname{in}_\omega(I)$, we can pick a finite set that still generates it. This gives the homog GB □

- The following result will be crucial:

Key Prop: $I \subset R$ homogeneous, $\omega \in \Gamma_{val}^{n+1}$, $g \in \mathrm{in}_\omega I$ homogeneous. Then $g = \mathrm{in}_\omega(f)$ for some $f \in I$.

Proof: Write $g = \sum_u c_u x^u \,\mathrm{in}_\omega(f_u)$ $f_u \in I$ homogeneous. $c_u \in \mathbb{k}\smallsetminus\{0\}$

$= \sum_u c_u \,\mathrm{in}_\omega(x^u f_u) = \sum_u c_u \,\mathrm{in}_\omega(\tilde{f}_u)$ $\tilde{f}_u \in I$ homogeneous $c_u \in \mathbb{k}$

Pick $a_u \in K$ $\overline{a_u} = c_u$, then $g = \sum_u \mathrm{in}_\omega(c_u \tilde{f}_u) = \sum_u \mathrm{in}_\omega(\tilde{\tilde{f}}_u)$
$\mathrm{val}(a_u)=0$

So we may assume $g = \sum_u \mathrm{in}_\omega(f_u)$ $f_u \in I$ homog, all of the same degree $= \deg(g)$

For each u, write $W_u = \overset{\text{finite}}{\mathrm{trop}}(f_u)(\omega). \in \Gamma_{val}$ because $\omega \in \Gamma_{val}^{n+1}$.

Set $f = \sum_u t^{W_u} f_u \in I$ homog of degree $= \deg(g)$. **Note**: $\mathrm{in}_\omega(t^{W_u} f_u) = \mathrm{in}_\omega(f_u)$

$\rightarrow$ this fails if $\omega \notin \Gamma_{val}^{n+1}$

Know: $\mathrm{trop}(f)_{(\omega)} \leq \max_u \mathrm{trop}(t^{W_u} f_u)_{(\omega)} = \max_u (\underbrace{-W_u + \mathrm{trop}(f_u)_{(\omega)}}) = 0$.

- If $\mathrm{trop}(f)_{(\omega)} < 0$, then $\sum_u \mathrm{in}_\omega(t^{W_u} f_u) = \sum_u \mathrm{in}_\omega(f_u) \overset{=0}{=} 0$ (it would have given the terms in f of weight $=0$, but there are none!)
  This implies $g = 0 = \mathrm{in}_\omega(0)$ ✓
- Otherwise, $\mathrm{trop}(f)_{(\omega)} = 0$ so $g = \sum_u \mathrm{in}_\omega(t^{W} f_u) \underset{\downarrow}{=} \mathrm{in}_\omega(\sum_u t^{W_u} f_u) = \mathrm{in}_\omega(f)$
  all terms have the same weight & it the weight of the sum of $t^{W_u} f_u$'s □

Natural questions: Fix $I \subset R$ homog ideal
$\omega \in \mathbb{R}^n \rightsquigarrow \mathrm{in}_\omega(I) \subseteq \mathbb{k}[x_0,\dots,x_n]$ ideal

Q1: How many initial ideals are there? Finitely many?

Q2: Given ω, ω', how to decide if $\mathrm{in}_\omega(I) = \mathrm{in}_{\omega'}(I)$?

Eg If $I = \langle g \rangle \rightsquigarrow$ Initial ideals $\longleftrightarrow$ dual complex to Newton subdivision of g
[monomials] $\longleftrightarrow$ max'l cells (ie $\omega \notin \mathcal{T}(V_{(g)})$)
So Q1 & Q2 have a nice answer for hypersurfaces (principal ideals).

Idea: $I \rightsquigarrow \mathrm{in}_\omega I$ is a (flat) degeneration & $\mathrm{in}_\omega I$'s are simpler to study.

- Towards Q2: need to iterate this construction, ie take initial ω-forms in $\mathbb{k}[x_0,\dots,x_n]$ where we have val($\mathbb{k}$) trivial (= "constant coefficient case")
  This allows us to study $\boxed{\text{how does } \mathrm{in}_\omega \text{ behave under small perturbations of } \omega.}$

Prop: Fix $f \in K[x_0,\dots,x_n]$ $\omega, v \in \mathbb{R}^{n+1}$. There exists $\varepsilon > 0$ such that $\forall \varepsilon'$ with $0 < \varepsilon' < \varepsilon$: $\mathrm{in}_v(\mathrm{in}_\omega(f)) = \mathrm{in}_{\omega + \varepsilon' v}(f)$.

Proof: Write $f = \sum_u c_u x^u$, $W = \operatorname{trop}(f)(\omega)$, so $\operatorname{in}_\omega f = \sum_{\substack{u \\ -\operatorname{val}(c_u)+\langle\omega,u\rangle = W}} \overline{c_u t^{-\operatorname{val}(c_u)}}\, x^u$

- By definition, $\operatorname{in}_v(\operatorname{in}_\omega f) = \sum'_{\substack{u\in\operatorname{supp}(\operatorname{in}_\omega f)\\ \langle u,v\rangle = \operatorname{trop}(\operatorname{in}_\omega f)(v)}} \overline{c_u t^{-\operatorname{val}(c_u)}}\, x^u$

Write $W' = \operatorname{trop}(\operatorname{in}_\omega f)(v) = \max_{u\in\operatorname{supp}(\operatorname{in}_\omega f)}\{-0 + \langle v,u\rangle\} = \max_{u\in\operatorname{supp}(\operatorname{in}_\omega f)}\{\langle v,u\rangle\}$

- $\operatorname{trop}(f)(\omega+\varepsilon' v) = \max_u\{-\operatorname{val}(c_u) + \langle \omega+\varepsilon' v, u\rangle\}$

$$= \max_u\{-\operatorname{val}(c_u) + \langle\omega,u\rangle + \varepsilon'\langle v,u\rangle\} \underset{(*)}{\leq} W + \varepsilon' W' \quad \text{if } \varepsilon' \ll 1$$

Proof of $(*)$: • If $u \in \operatorname{supp}(\operatorname{in}_\omega f)$, this holds trivially:

$$-\operatorname{val}(c_u) + \langle\omega,u\rangle + \varepsilon'\langle v,u\rangle = W + \varepsilon'\langle v,u\rangle \leq W + \varepsilon' W'$$
& = for some u.

• If $u \notin \operatorname{supp}(\operatorname{in}_\omega f)$, then $-\operatorname{val}(c_u) + \langle\omega,u\rangle < W$ & we can find $\varepsilon' \ll 1$ with $-\operatorname{val}(c_u) + \langle\omega,u\rangle + \underbrace{\varepsilon'\langle v,u\rangle} < W + \varepsilon' W'$.

Furthermore, the previous argument shows that for some u in $\operatorname{supp}(\operatorname{in}_\omega f)$, equality in $(*)$ is achieved, so $\operatorname{trop}(f)(\omega+\varepsilon' v) = W + \varepsilon' W'$ for $\varepsilon' \ll 1$. Moreover,

$\{u : -\operatorname{val}(c_u) + \langle\omega+\varepsilon' v, u\rangle = W + \varepsilon' W'\} = \{u : -\operatorname{val}(c_u) + \langle\omega,u\rangle = W \ \&\ \langle v,u\rangle = W'\}$ for $\varepsilon' \ll 1$.

We can find $\varepsilon > 0$ so that these sets are $=$ for any $\varepsilon' < \varepsilon$ & by definition

$$\operatorname{in}_v(\operatorname{in}_\omega f) = \operatorname{in}_{\omega+\varepsilon' v}(f) \qquad \forall\, 0<\varepsilon'<\varepsilon. \quad \square$$

Note: In later lectures we'll see that this proposition extends from (Laurent) polynomials to homogeneous ideals.