Lecture IX: Gröbner basis over valued fields III

Recall: Given $I \subset K[x_0,\dots,x_n] = R$ homogeneous, $w \in \mathbb{R}^{n+1}$ we showed that

- $\mathrm{in}_w(I) \subseteq \Bbbk[x_0,\dots,x_n]$ is homog.
- I admits a homogeneous Gröbner basis for w (finite set $\{g_1,\dots,g_s\} \subseteq I$ where $\mathrm{in}_w I = \langle \mathrm{in}_w(g_1), \dots \mathrm{in}_w(g_s)\rangle$)

If $w \in \Gamma_{val}^{n+1}$ & $g \in \mathrm{in}_w I$ homog, then $g = \mathrm{in}_w(f)$ for some $f \in I$ homog.

Lemma 0: If $w \in \mathbb{R}^n$, $x^a \in \mathrm{in}_w I$ homog of degree d, then $g = \mathrm{in}_w(f)$ $f \in I$ homog of deg d. ($f \in I_d$)

Pf/ Write $x^a = \sum_u \mathrm{in}_w(f_u)$ for $f_u \in I_d$ $\quad (*)$

Given v, v' with $\mathrm{supp}\, \mathrm{in}_w(f_v) \cap \mathrm{supp}\, \mathrm{in}_w(f_{v'}) \neq \emptyset$, say $x^u \in \mathrm{supp}\, \mathrm{in}_w f_v \cap \mathrm{supp}\, \mathrm{in}_w f_{v'}$.

Write $f_v = \sum_p c_{vp} x^p$, $\quad f_{v'} = \sum_p c_{v'p} x^p$ $\quad c_{vu}, c_{v'u} \neq 0$.

Then $\mathrm{trop}(f_v)(w) = -\mathrm{val}(c_{vu}) + \langle w, u\rangle$
$\quad\;\;\mathrm{trop}(f_{v'})(w) = -\mathrm{val}(c_{v'u}) + \langle w, u\rangle$

so $W' = \mathrm{trop}(f_v)_{(w)} - \mathrm{trop}(f_{v'})_{(w)} = -\mathrm{val}(c_{vu}/c_{v'u}) \in \Gamma_{val}$

Then $\mathrm{in}_w(f_v) + \mathrm{in}_w(f_{v'}) = \mathrm{in}_w(f')$ for $f' = f_v + t^{W'} f_{v'} \in I_d$ $\quad (**)$

[The identity follows by the same proof as Key Prop from Lecture VIII.]

We join f_v's with non disjoint support (by induction on the # of summands in $(*)$) as in $(**)$ until all summands have pairwise disjoint support. Then, by Lemma 1 from Lecture VIII we can write $x^a = \mathrm{in}_w(f_u)$ (no cancellations can occur & x^u lies in the support of one …) $\square$

- Want to study the structure of $\mathbb{R}^n$ determined by w's giving the same initial ideals.

Q1: How does $\mathrm{in}_w I$ behave under small perturbations of w?

Prop 1 (Lecture VIII) Given $f \in R$ & $w, v \in \mathbb{R}^{n+1}$ there exists $\varepsilon > 0$ such that
$$\mathrm{in}_v(\mathrm{in}_w f) = \mathrm{in}_{w+\varepsilon' v}(f) \quad \forall\, 0 < \varepsilon' < \varepsilon.$$

- We extend this Prop to homogeneous ideals:

Lemma 1: Let $I \subset R$ homog & $w \in \mathbb{R}^{n+1}$. Then, we can find $v \in \mathbb{R}^{n+1}$ & $\varepsilon > 0$ such that:

(i) $\mathrm{in}_v(\mathrm{in}_w I)$ & $\mathrm{in}_{w+\varepsilon v}(I)$ are homogeneous ideals in $\Bbbk[x_0,\dots,x_n] = \widetilde{R}$

(ii) $\mathrm{in}_v(\mathrm{in}_w I) \subseteq \mathrm{in}_{w+\varepsilon v}(I)$

Proof: See the end of the notes.

Next goal: show the statement holds $\forall\, \varepsilon' < \varepsilon$ & $=$ holds in (ii). To this end, we need to measure how the ideals are changing ⇝ Hilbert functions.

Given J homog ideal in R (or $\tilde{R}$) : $\mathrm{Hilb}_J : \mathbb{N} \longrightarrow \mathbb{N}$, $d \longmapsto \dim (R/J)_d$

Note: • R/J is a K-graded algebra & each piece is a finite dim'l K-v.s.

• For $d >> 0$, $\mathrm{Hilb}_J(d) = \mathrm{HilbPoly}(d)$ & degree (Hilbert polynomial) $= \mathrm{Krull\,dim}(R/J) - 1$.

Lemma 2: Fix $I \subset R$ homog, $d \in \mathbb{N}$. Assume $w \in \mathbb{R}^{n+1}$ satisfies that $(\mathrm{in}_w I)_d$ is spanned over $\mathbb{K}$ by its monomials (equiv for $\mathrm{in}_w I_d$) Then:

$$B_d = \{ x^u : |u| = d, \ x^u \notin \mathrm{in}_w I \}$$

gives a K-basis for $(R/I)_d$.

Proof: • Claim 1: $B_d \subset (R/I)_d$ is K-linearly independent

Pf/ If not, $\exists$ f homog of degree d, with $f = \sum_{u \in B_d} c_u x^u \in I$ & not all $c_u = 0$.

$\Rightarrow$ $\mathrm{in}_\omega(f)$ has no monomial in $(\mathrm{in}_w I)_d$ (by hypothesis) $(= \mathrm{in}_w I_d)$, but $\mathrm{in}_w(f) \in \mathrm{in}_w I$, Contradiction! □

We have • $|B_d| + \dim_{\mathbb{K}}(\mathrm{in}_w I_d) = \binom{n+d}{n}$

• $\dim_K I_d + \dim_K (R/I)_d = \binom{n+d}{n}$ (**)

From Claim 1: $\dim_K (R/I)_d \geqslant |B_d|$, hence $\boxed{\dim_{\mathbb{K}} (\mathrm{in}_w I)_d \geqslant \dim_K I_d}$

• Claim 2: Given x^u in $(\mathrm{in}_w I)_d$, $\exists f_u \in I_d$ with $\mathrm{in}_w f_u = x^u$ (use Lemma 0)

• Claim 3: $\mathcal{G} = \{ f_u : x^u \in (\mathrm{in}_w I)_d \}$ is linearly independent in R

Pf/ If they were l.d., write $\sum_u a_u f_u = 0$ with not all $a_u = 0$.

Write $f_u := x^u + \sum c_{uv} x^v$ for f_u in $\mathcal{G}$. (1)

• Since $\mathrm{in}_w f_u = x^u$, we know $\boxed{-\mathrm{val}(c_{uv}) + \langle w, v \rangle < \langle w, u \rangle \quad \forall v \neq u}$ $\forall u$

• $0 = \sum_u a_u f_u = \sum_u (a_u + \sum_{v \neq u} a_v c_{vu}) x^u$ so $a_u + \sum_{v \neq u} a_v c_{vu} = 0 \ \forall u$

• Pick u' realizing $\mathrm{trop}(\sum_u a_u x^u)(w) = \max_u \{ -\mathrm{val}(a_u) + \langle w, u \rangle \}$ (*)

Then $0 = a_{u'} + \sum_{v \neq u'} a_v c_{vu'}$ gives $a_{u'} = \sum_{v \neq u'} -a_v c_{vu'}$ so

$\mathrm{val}(a_{u'}) \geqslant \mathrm{val}(a_v) + \mathrm{val}(c_{vu'})$ for some $v \neq u'$ ($\mathrm{val}(a+b) \geqslant \min\{v(a), v(b)\}$)

Hence: $-\mathrm{val}(a_v) - \mathrm{val}(c_{vu'}) + \langle w, u' \rangle \geqslant -\mathrm{val}(a_{u'}) + \langle w, u' \rangle \underset{\text{by }(*)}{\geqslant} -\mathrm{val}(a_v) + \langle w, v \rangle$ $\forall v \neq u'$.

So $\boxed{-\mathrm{val}(c_{vu'}) + \langle w, u' \rangle \geqslant \langle w, v \rangle \quad \text{for } v \neq u'}$ (2)

(1) & (2) contradict each other! We conclude $\mathcal{G}$ is lin indep □.

Claim 3 $\Rightarrow$ $\boxed{\dim_K I_d \geq \dim_{\mathbb{K}} (\mathrm{in}_w I)_d}$ (xxx)

From (xx) & (xxx) we get $\dim_K I_d = \dim_{\mathbb{K}} (\mathrm{in}_w I)_d$ & so
$\dim_K (R/I)_d = |\mathcal{B}_d|$

From Claim 1, $\mathcal{B}_d$ is K-lin indep in $(R/I)_d$ so it's a K-basis for $(R/I)_d$ $\square$

- This Lemma shows that Hilbert functions are preserved under taking initials:

<u>Corollary 1</u>: For any $w \in \mathbb{R}^{n+1}$ & $I \subset R$ homogeneous ideal, then:

$$\mathrm{Hilb}_I = \mathrm{Hilb}_{\mathrm{in}_w I} \quad \text{, ie} \quad \dim_K (R/I)_d = \dim_{\mathbb{K}} (\tilde{R}/\mathrm{in}_w I)_d \quad \forall d \geq 0$$

In particular the rings R/I & $R/\mathrm{in}_w I$ have the same Krull dimension.

<u>Proof</u>: By Lemma 1, $\exists\, v \in \mathbb{R}^{n+1}$ & $\varepsilon > 0$ satisfying $\mathrm{in}_v(\mathrm{in}_w I) \subseteq \mathrm{in}_{w+\varepsilon v} I$ & both are monomial ideals. Both are homog, so $\forall d$: $(\mathrm{in}_v(\mathrm{in}_w I))_d \subseteq (\mathrm{in}_{w+\varepsilon v} I)_d$.

For $w' = w + \varepsilon v$ & $d \geq 0$ we know that $(\mathrm{in}_{w'} I)_d$ is spanned by its monomials, so we can apply Lemma 2 to conclude that the monomials of deg d not in $(\mathrm{in}_{w'} I)_d$ are a K-basis of $(R/I)_d$. So $\boxed{\dim_{\mathbb{K}} (R/I)_d = \dim_{\mathbb{K}} (\tilde{R}/\mathrm{in}_{w'} I)_d}$ (*)

- <u>Claim</u>: $\mathrm{in}_v(\mathrm{in}_w I) = \mathrm{in}_{w+\varepsilon v}(I)$. (Equiv, for any $d \geq 0$ $(\mathrm{in}_v(\mathrm{in}_w I))_d = (\mathrm{in}_{w+\varepsilon v} I)_d$)

Pf/ Assume the contrary & pick $x^u \in (\mathrm{in}_{w+\varepsilon v} I)_d \smallsetminus (\mathrm{in}_v(\mathrm{in}_w I))_d$ for some $d \geq 0$.

Notice that by Lemma 0, we have $x^u = \mathrm{in}_{w+\varepsilon v}(f_u)$ for some $f_u \in I_d$.

Write $f_u = x^u + f'_u$ with $f'_u \in R_d$ has <u>all</u> its monomials outside $(\mathrm{in}_{w+\varepsilon v} I)_d$

So f'_u has no monomial in $(\mathrm{in}_v(\mathrm{in}_w I))_d$.

We conclude: f_u has no monomial in $(\mathrm{in}_v(\mathrm{in}_w I))_d$ so $\mathrm{in}_v(\mathrm{in}_w f_u) \notin \mathrm{in}_v(\mathrm{in}_w I)_d$

But $f_u \in I$ homog of degree d, a contradiction! $\square$

- From the Claim we know that $v \in \mathbb{R}^{n+1}$ satisfies $(\mathrm{in}_v(\mathrm{in}_w I))_d$ is monomial, $\forall d$, & $d \geq 0$ hence spanned by its monomials. By Lemma 2 applied to $\mathrm{in}_w I$ over $\mathbb{K}$ we conclude that

$$\dim_{\mathbb{K}} (\tilde{R}/\mathrm{in}_w I)_d = \dim_{\mathbb{K}} (\tilde{R}/\mathrm{in}_v(\mathrm{in}_w I))_d = \dim_{\mathbb{K}} (\tilde{R}/\mathrm{in}_{w+\varepsilon v} I)_d$$
$$= \dim_K (R/I)_d \quad \text{by (*)}$$

The result holds for all d.

Corollary 2: $I \subset R$ homog. For any w, v in $\mathbb{R}^{n+1}$, there exists $\varepsilon > 0$ s.t.

$$\mathrm{in}_v(\mathrm{in}_w I) = \mathrm{in}_{w+\varepsilon' v}(I) \qquad \forall\, 0 < \varepsilon' < \varepsilon.$$

Proof: By Noetherianness, we can find $\{g_1, \ldots, g_s\} \subseteq \mathbb{K}[x_0, \ldots, x_n] = \tilde{R}$ a generating set for $\mathrm{in}_v(\mathrm{in}_w I)$ where each $g_i = \mathrm{in}_v(\mathrm{in}_w(f_i))$ for some $f_i \in I$.

By Prop 1 (Lecture VIII) $\exists\, \varepsilon > 0$ with $g_i = \mathrm{in}_v(\mathrm{in}_w(f_i)) = \mathrm{in}_{w+\varepsilon' v}(f_i)$ $\forall i$ $\forall 0<\varepsilon'<\varepsilon$

Then $\boxed{\mathrm{in}_v(\mathrm{in}_w I) \subseteq \mathrm{in}_{w+\varepsilon' v}(I)}$ $(*)$

By Corollary 1 : • $\mathrm{in}_v(\mathrm{in}_w I)$ has the same Hilb functions as $\mathrm{in}_w I$.

• $\mathrm{in}_w I$ & $\mathrm{in}_{w+\varepsilon' v} I$ have ——— $\mathrm{in}_w I$.

So $\mathrm{in}_v(\mathrm{in}_w I)$ & $\mathrm{in}_{w+\varepsilon' v} I$ have the same Hilb functions. By $(*)$, they must be the same ideal. $\square$

• Typical applications : I irreducible. However $\mathrm{in}_w I$ is rarely irreducible.

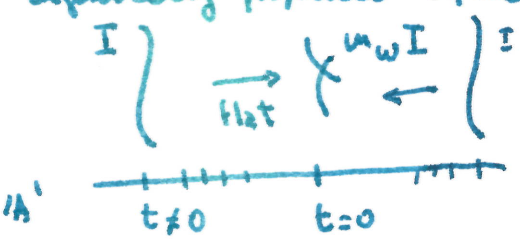
Q: What can we say regarding each of its irred. components? $\Leftrightarrow$ Minimal associated primes

Def: $J \subseteq P$, P prime & if $J \subsetneq P' \subseteq P$ for P' prime then $P' = P$ (P min assoc prime of J)

Lemma 3 $I \subset R$ homog prime ideal of $\dim = d$ & $w \in \Gamma_{val}^{n+1}$. Then, every minimal associated prime of $\mathrm{in}_w I$ has dimension $= d$.

Proof: See [MS Lemma 2.4.12] Prove it for $w = 0 \in \mathbb{R}^{n+1}$!

• Key geometric property : Gröbner bases for term orders give models over $\mathbb{A}^1$ with good degeneracy properties (flatness!)



(In particular: dimension is the same along ≠ fibers)

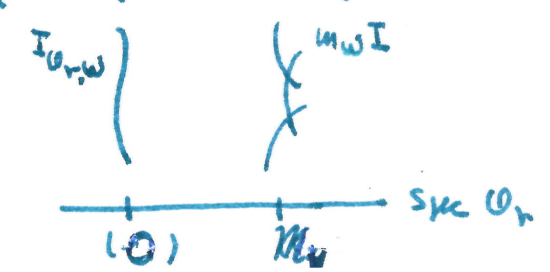
This property holds when K has a valuation & $w \in \Gamma_{val}^{n+1}$. We get a family over $\mathrm{Spec}(\mathcal{O}_v) = \{P \in \mathcal{O}_v : \text{prime}\}$. $\mathcal{O}_v$ has either • no prime $\neq 0$ ($\Gamma_{val} = \{1\}$)

• 2 primes : (0) & $\mathfrak{m}_v$ ($\Gamma_{val} \neq \{1\}$)

• Fiber over (0) := special fiber, fiber over $\mathfrak{m}_v$:= generic fiber

CASE 1 : $\Gamma_{val} = \{1\}$ $\quad K = \mathcal{O}_v = \mathbb{K}$ & special fiber = generic fiber

CASE 2 :



$I_{\mathcal{O}_v, w} = \langle \tilde{f}(x) : f \in I \rangle \subseteq \mathcal{O}_v[x_0, \ldots, x_n]$

$\tilde{f}(x) = t^{\mathrm{trop}(f)(w)} f(t^{-w_0} x_0, \ldots, t^{-w_n} x_n)$

$w = (w_0, \ldots, w_n) \in \Gamma_{val}^{n+1}$.

• Why $\tilde{f}$? $f = \sum c_u x^u \rightsquigarrow f(t^{-w_0}x_0, \ldots, t^{-w_n}x_n) = \sum_u c_u t^{-\langle w,u\rangle} x^u$

$\Rightarrow \operatorname{trop}(f)_{(w)} + \operatorname{val}(c_u t^{-\langle w,u\rangle}) = \operatorname{val}(c_u) - \langle w,u\rangle + \operatorname{trop}(f)_{(w)}$
$= \operatorname{trop}(f)_{(w)} - (-\operatorname{val}(c_u) + \langle w,u\rangle) \geq 0$

Then $\tilde{f} \in \mathcal{O}_r[\underline{x}]$. Furthermore, $\operatorname{in}_w(\tilde{f}) = \operatorname{in}_w(f)$.

Lemma 4: $I \subset R$ homog, $w \in \Gamma_{\mathrm{val}}^{n+1}$, $\Gamma_{\mathrm{val}} \neq \{0\}$
$M = \mathcal{O}_r[\underline{x}]/I_{\mathcal{O}_r,w}$ is a flat $\mathcal{O}_r$-module with $M \otimes K \simeq K[\underline{x}]/I$

& $M \otimes_{\mathcal{O}_r} \mathbb{k} \cong \mathbb{k}[x_0 \ldots x_n]/\operatorname{in}_w I$.

So $\operatorname{Spec} M \longrightarrow \operatorname{Spec} \mathcal{O}_r$ is a flat family with

- generic fiber $\simeq \operatorname{Spec}(K[x]/I)$
- special fiber $\simeq \operatorname{Spec}(\mathbb{k}[\underline{x}]/\operatorname{in}_w I)$

Proof: See end of notes.

Remark: Assumption $w \in \Gamma_{\mathrm{val}}^{n+1}$ is hard to remove. Ex, if K has trivial valuation, then for generic $w \in \mathbb{R}^{n+1}$: $\operatorname{in}_w I$ is monomial
But generic & special fibers agree, and I need not be monomial!
(because $\mathbb{k} = K = \mathcal{O}_r$)

Proof of Lemma 4:

- **Flatness**: M is flat over $\mathcal{O}_r$ if $0 \to N' \to N$ N, N' R-modules
  Then $M \otimes_{\mathcal{O}_r} -$ is exact: $0 \longrightarrow M \otimes_R N' \longrightarrow M \otimes_R N$ is exact.

By Eisenbud's Comm Algebra [Prop 6.1] M is flat $\iff$ $\operatorname{Tor}_1^R(\mathcal{O}_r/I, M) = 0$ $\forall I \subset \mathcal{O}_r$ f.g. ideal

Note: If $I = \langle a \rangle$, then $\operatorname{Tor}_1^R(R/(a), M) = (0:_M x) = \{m \in M : xm = 0\}$.

- **Claim**: $\mathcal{O}_r$ is a valuation ring, hence every finitely generated ideal is principal.

Proof: $I = \langle g_1, \ldots, g_s\rangle$ $\operatorname{val}(g_1) \leq \operatorname{val}(g_2) \leq \cdots \leq \operatorname{val}(g_s)$ so $I = \langle g_1 \rangle$
because $a_i = \frac{a_i}{a_1} a_1$ & $\operatorname{val}(a_i/a_1) \geq 0$ so $\frac{a_i}{a_1} \in \mathcal{O}_r$. $\forall i$. ∎

- $\operatorname{Tor}_1^{\mathcal{O}_r}(\mathcal{O}_r/(a), M) = 0 \iff a$ is a nonzero divisor in M.

Claim: M is torsion free.

Pf/ Note that $M = \mathcal{O}_r[x]/I_{\mathcal{O}_r,w} \underset{\substack{\mathcal{O}_r\text{-alg.}\\ \text{iso}}}{\simeq} R^w/I^w$ where $R^w = \{\sum c_u x^u : -\operatorname{val}(c_u) + \langle w,u\rangle \leq 0 \ \forall u\} \subset K[x]$

$\tilde{x}_i \longmapsto t^{w_i} x_i$ $I^w = I \cap R^w$.

Then, if $f \in R^w$ with $af \in I^w$, then $af \in I$, so $f \in I$, hence $f \in I^w$ $\square$

(1). Special fiber $:= \Pi \otimes_{\mathcal{O}_v} \mathcal{O}_v/m_K = \Pi \otimes_{\mathcal{O}_v} \mathbb{K}$

(2). General fiber $:= \Pi \otimes_{\mathcal{O}_v} (\mathcal{O}_v/_{(0)})_{(0)} = \Pi \otimes_{\mathcal{O}_v} K$.

(2). **Claim**: $\Psi: R^w/I^w \otimes_{\mathcal{O}_v} K \longrightarrow R/I$ is an iso.

$f \otimes a \longmapsto af$ (& extend linearly on each comp.)

Pf/ Well defined: $f \in I^w$, then $\Psi(f \otimes 1) = f \in I$ by definition ✓

- Injective: $\Psi\left(\sum_{i \in \Lambda} \mu_i (f_i \otimes a_i)\right) = 0$ with $\mu_i \in \mathbb{Z}$ $f_i \in R^w$, $a_i \in K^\times$ $\forall i$, Λ: finite

ie $\sum_i \mu_i f_i a_i \in I$

Take $v = \min\{val(a_i) : i \in \Lambda\}$. By definition, $f_i \otimes a_i = \frac{f a_i}{t^v} \otimes t^v$ because $a/t^v \in \mathcal{O}_v$.

Then $\sum \mu_i f_i \otimes a_i = \left(\sum_i \mu_i \frac{f_i a_i}{t^v}\right) \otimes t^v = \left(t^{-v} \sum_i \mu_i f_i a_i\right) \otimes t^v$

- Since $\sum_i \mu_i f_i a_i \in I$, so does $t^{-v} \sum_i \mu_i f_i a_i$
- $t^{-v} \sum \mu_i f_i a_i \in R^w$

$\Big\}$ $t^{-v} \sum_i \mu_i f_i a_i \in I^w$

We conclude $\sum_{i \in \Lambda} \mu_i (f_i \otimes a_i) = 0$ in R^w/I^w, so Ψ is injective.

- Surjective: Pick $a_u x^u \in R$ & $c \in K$ with ~~$-val(c) + \langle w, u \rangle$~~ ≤ 0 ($\exists$ because $\Gamma_{val} \neq \{0\}$)

$\Psi\left(c x^u \otimes \frac{a_u}{c}\right) = a_u x^u$ $\quad c x^u \in R^w$, $\frac{a_u}{c} \in K$.

(1) **Claim**: $\varphi: \mathcal{O}_v[x]/I_{\mathcal{O}_v, w} \otimes_{\mathcal{O}_v} \mathbb{k} \longrightarrow \mathbb{K}[x]/in_w I$ is a iso

$(f \otimes c) \longmapsto \overline{\overline{fc}}$ $\quad c \in \mathbb{K} = \mathcal{O}_v/m_v$ & extend linearly

from $\mathcal{O}_v[x]$ to $\mathbb{K}[x]$

Pf/ φ is well-defined: If $f \in I$ & $\tilde{f} \in J_{\mathcal{O}_v, w}$, then $\varphi(\tilde{f} \otimes 1) = in_w(f)$ (see page 5)

- φ surjective by construction.
- φ injective:

Step 1: $f \in \mathcal{O}_v[x]$ & $\overline{f} = 0$, then $\overline{f} \in in_w(I)$ Break f into its homogeneous components, $f = \sum_i f_i$ & $\overline{f} \in in_w(I) \Rightarrow f_i \in in_w(I)$ & each $f_i = in_w(h_i)$ for some $h_i \in I$ because $w \in \Gamma_{val}^{n+1}$ (Lecture VIII)

view f in $\mathbb{K}[x]$

Consider $\tilde{h}_i$ for each i: $\tilde{h}_i \in \mathcal{O}_v[x]$. & $h = \sum_i \tilde{h}_i \in I_{\mathcal{O}_v,\omega}$

Then $\overline{\tilde{h}_i} = m_\omega(h_i) = f_i \quad \forall i$, so $\overline{h} = \overline{F}$ in $\Bbbk[x]$, ie $h - F \in \mathfrak{m}_v \mathcal{O}_v[x]$

<u>Step 2</u>: $\Psi\left(\sum_i m_i (f_i \otimes a_i)\right) = 0$ for $m_i \in \mathbb{Z}$, $f_i \in \mathcal{O}_v[\underline{x}]$, $a_i \in \mathcal{O}_v$.

So $\sum m_i (f_i \otimes a_i) = \left(\sum_i m_i f_i a_i\right) \otimes 1 \longmapsto 0$ implies $\overline{\sum m_i a_i f_i} \in m_\omega f$

By Step 1, we have $\exists\, h \in I_{\mathcal{O}_v,\omega}$, & $\sum m_i a_i f_i - h \in \mathfrak{m}_v \mathcal{O}_v[x]$, say

$\sum_i m_i a_i f_i = h + a h_1 \qquad h, h_1 \in \mathcal{O}_v[\underline{x}]$, $a \in \mathfrak{m}_v$ (coeff in the expression form a f.g. ideal in $\mathcal{O}_v$, contained in $\mathfrak{m}_v$, hence they are multiples of a single element $a \in \mathfrak{m}_v$.

Then $\sum m_i a_i f_i \otimes 1 = (h + a h_1) \otimes 1 = \underset{\substack{\parallel \\ 0\ (h \in I_{\mathcal{O}_v,\omega})}}{h \otimes 1} + h_2 \otimes \underset{\substack{\cap \\ \mathfrak{m}_v}}{a} = 0$ in $\dfrac{\mathcal{O}_v[x]}{I_{\mathcal{O}_v,\omega}} \otimes_{\mathcal{O}_v} \Bbbk$.

$\square$

<u>Proof of Lemma 1</u>:

- Start by finding σ with $m_v(m_\omega I)$ monomial.