Lecture X: Gröbner complexes

Recall: $I \subset K[x_0, \dots, x_n] = R$ homogeneous ideal

$w \in \mathbb{R}^{n+1} \rightsquigarrow in_w(I) \subseteq K[x_0, \dots, x_n]$ ideal

Q1: How many initial ideals are there? Finitely many? How many are monomial / contain a mon?

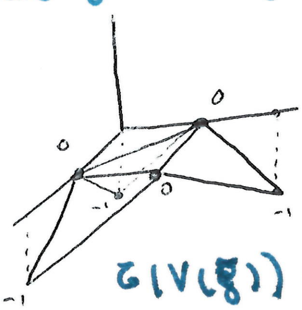
Q2: given w, w': how to decide if $in_w(I) = in_{w'}(I)$.

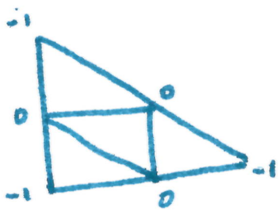
GOAL: Study the (polyhedral) subdivision of $\mathbb{R}^{n+1}$ induced by ("cells $\leftrightarrow$ same initials")

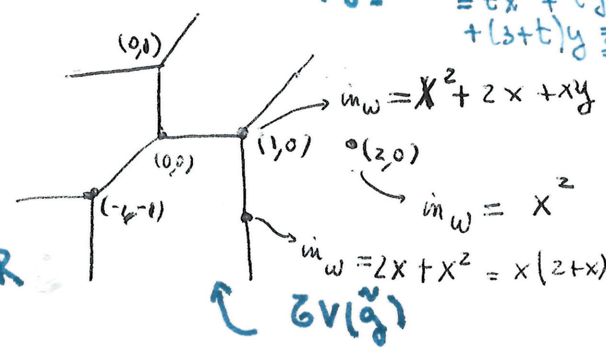
Eg: $I = \langle g \rangle$ then subdivision is the (regular) Newton subdivision of g

$\mathcal{T}V(g)$ = union of codim 1 cells, max cells $\longleftrightarrow$ $in_w(g)$ = monomial

Eg: $\tilde{g} = t x^2 + t y^2 + t^2 + t^2 x + (3+t) y + xy + t \in \mathbb{C}\{\{t\}\}[x,y]$ $\rightsquigarrow g = g^{hom}$ (w.r.t. z)
$= t x^2 + t y^2 + (2+t) x z + (3+t) y z + xy + t z^2$







$\mathcal{T}(V(g)) = (\mathcal{T}(V(\tilde{g})) \times \{0\}) \times \mathbb{R}$

Claim:
- 6 monomial $in_w \leftrightarrow$ 6 chambers
- 3 $in_w I$ with 3 terms (vertices of $\mathcal{T}(V(\tilde{g}))$)
- 9 " " 9 " (edges of $\mathcal{T}(V(g))$)

Last 2 lectures: studied how does $in_w I$ behave under small perturbations of w

Coro 1: For any $w, v \in \mathbb{R}^{n+1}$, $I \subset R$ homogeneous. Then:

(1) $\text{Hilb}_I = \text{Hilb}_{in_w I}$ & (2) $\exists \varepsilon > 0$: $in_v(in_w I) = in_{w+\varepsilon' v}(I)$ $\forall 0 < \varepsilon' < \varepsilon$.

$\mathbb{N} \ni d \mapsto \dim_K (R/I)_d$

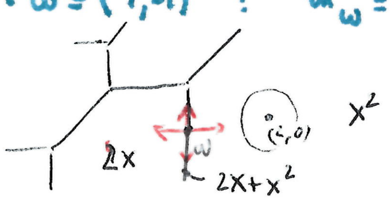
- In the example above: $I = \langle g \rangle$ gen in degree 2 (in $K[x,y,z]$)

Take $w = (2,0)$: $in_w I = x^2$ so $\text{Hilb}_I = \text{Hilb}_{in_w I} = ($ $d = 0, 1, 2, \dots, d$: $1, 3, 5, 7, 9, \dots, 1+2d, \dots)$

xy, y^2, xz, yz, z^2 ; $xy^2, y^3, \dots$; $\{x y^i z^j, y^a z^b\}$, $i+j = d-1$, $a+b = d$
$= \binom{d}{1} + \binom{1+d}{1} \Rightarrow [1+2d]$

- $w = (2,0)$ for any v, $\varepsilon = \frac{1}{2}$ works

- $w = (1,-1)$: $in_w = 2x + x^2$ if $v \notin \langle (0,1) \rangle$, then $in_{w+\varepsilon v} = \begin{cases} x^2 \\ x \end{cases}$



if $v \in \langle (0,1) \rangle$, then for $\varepsilon \ll 1$ $in_{w+\varepsilon v} = in_w$

$in_v(2x + x^2) = 2x + x^2$

We aim for a similar behavior for higher codimension ideals!

<u>Definition</u>: Given $w \in \mathbb{R}^{n+1}$, we set $C_I[w] := \{ w' \in \mathbb{R}^{n+1} : in_{w'} I = in_w I \}$

The closure in the Euclidean topology $\overline{C_I[w]} \subseteq \mathbb{R}^{n+1}$ will be our building blocks.

<u>Example above</u>: $C_I((2,0)) =$  $= \{ -1 + 2x \geq \max\{-1+2y, x, y, x+y\}\}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathrm{trop}(g - tx^2)_{(w)}$

<u>Lemma 1</u>: $\overline{C_I[w]} \ni w'$, then $w + \mathbb{R}\underline{1} \in \overline{C_I[w]}$, so the line $\mathbb{R}\underline{1}$ is in the lineality space of $\overline{C_I[w]}$. $(\underline{1} = (1, \ldots, 1))$

<u>Proof</u>: $in_{w'} I$ is generated by $\{ in_{w'}(f) \; f \in I \text{ homog}\}$ & since f is homog:

$f = \sum_u c_u x^u \rightsquigarrow \mathrm{trop}(f)_{(w)} = \max_u \{ -\mathrm{val}(c_u) + \langle w, u \rangle \}$

$\mathrm{trop}(f)_{(w + \lambda \underline{1})} = \max_u \{ -\mathrm{val}(c_u) + \langle w, u \rangle + \lambda \underbrace{\langle \underline{1}, u \rangle}_{\deg f} \}$

$= \mathrm{trop}(f)_{(w)} + \lambda \deg f.$

so $in_{w'}(f) = in_{w' + \lambda \underline{1}}(f)$. $\forall \lambda \in \mathbb{R}$

Since $w' \in C_I[w]$ implies $w' + \mathbb{R}\underline{1} \subseteq C_I[w]$, the same holds after taking closure in $\mathbb{R}^{n+1}$ $\square$

<u>Prop 1</u>: The set $\overline{C_I[w]}$ is a Γ_{val}-rational polyhedron whose lineality space contains $\underline{1}$. If $in_w I$ is not a monomial ideal, then $\exists\, \hat{w}' \in \mathbb{R}^{n+1}$ such that $in_{\hat{w}'}(I)$ is a monomial ideal and $\overline{C_I[w]}$ is a proper face of $\overline{C_I[\hat{w}']}$.

<u>Proof</u>: Lemma 1 from Lecture IX: $\exists\, v \in \mathbb{R}^{n+1}$ with $in_v(in_w I)$ a monomial ideal $= in_{w + v\varepsilon}(I)$ for $\varepsilon \ll 1$ [By Cor 2 L IX]

<u>STEP 1</u>: Find monomial initial ideals $\geq C_I[w]$.

Take $\tilde{w} = w + \varepsilon v \in \mathbb{R}^{n+1}$, write $in_{\tilde{w}} I = \langle x^{u_1}, \ldots, x^{u_s} \rangle$ $\quad d_i = |u_i| \;\; \forall i = 1, \ldots, s$

Lemma 2 Lecture IX: monomials deg d_i not in $(in_{\tilde{w}} I)_{d_i}$ are a basis of $(R/I)_{d_i}$

$\Rightarrow$ Can write x^{u_i} in these bases: $g_i = x^{u_i} + \underbrace{\sum c_{iv} x^v}_{= g_i'} \in I$ $\quad x^v \notin in_{\tilde{w}} I$. $\forall c_{iv} \neq 0$.

- No monomial of g_i' lies in $(in_{\tilde{w}} I)_{d_i} \otimes K$.
- $in_v(in_w(g_i)) \in in_{\tilde{w}} I$, so $in_v(in_w(g_i)) = x^{u_i} \Rightarrow in_{\tilde{w}}(g_i) = x^{u_i}$

<u>Conclusion</u>: $\{g_1, \ldots, g_s\}$ is a GB for I with resp. to $\tilde{w}$.

<u>Claim</u>: $\overline{C_I[\tilde{w}]} = \{ z \in \mathbb{R}^{n+1} : u_i \cdot z \geq -\mathrm{val}(c_{iv}) + v \cdot z \quad \forall\, 1 \leq i \leq s \;\; |v| = d_i \}$

$(\therefore \Gamma_{\mathrm{val}}\text{-polyhedron})$

3F/ (⊆) Assume the contrary, $w' \in C_I[\tilde{w}]$ but for some i, v we have $u_i \cdot z < -\mathrm{val}(c_{i,v}) + v \cdot z$. In particular: $\mathrm{in}_{w'}(g_i) \neq x^{u_i}$.

But $\mathrm{in}_{\tilde{w}} I = \mathrm{in}_{w'} I$ (by def) is a monomial ideal, every term of $\mathrm{in}_{w'}(g_i)$ lies in $\mathrm{in}_{w'} I$. But this contradicts the construction of g_i (none of its mons, except x^{u_i} lie in $\mathrm{in}_{\tilde{w}} I$) so $C_I[\tilde{w}] \subseteq$ (RHS) & we conclude by taking closure (RHS is closed).

(⊇) Prove rel int(RHS) $\subseteq C_I[\tilde{w}]$. Assume w' satisfies $u_i \cdot w' > -\mathrm{val}(c_{i,v}) + v \cdot w'$ $\forall i$

Then $\mathrm{in}_{w'}(g_i) = x^{u_i}$ $\forall i$ so $\mathrm{in}_{\tilde{w}}(I) \subseteq \mathrm{in}_{w'}(I)$. Both ideals have the same Hilbert function ($= \mathrm{Hilb}_I$) so $=$ holds & $w' \in C_I[\tilde{w}]$. Conclude by taking closure in $\mathbb{R}^{n+1}$. □

• <u>STEP 2</u> $\overline{C_I[w]}$ is a Γ_{val}-rational polyhedron

<u>Claim 2</u>: $C_I[w] \subseteq \overline{C_I[\tilde{w}]}$ (exercise)

<u>Claim 3</u>: $\overline{C_I[w]}$ is a face of $C_I[\tilde{w}]$ ($\Rightarrow \Gamma_{val}$-rat'l polyhedron)

<u>Proof</u>: By construction, $\{\mathrm{in}_w(g_1), \ldots, \mathrm{in}_w(g_s)\}$ is a GB for $\mathrm{in}_w(I)$ with respect to v

<u>Note</u>: If $\mathrm{in}_{w'}(I) = \mathrm{in}_w(I) \Rightarrow \mathrm{in}_{w'}(g_i) = \mathrm{in}_w(g_i)$ $\forall i$.

Why? $x^{u_i} \in \mathrm{supp}\, \mathrm{in}_{w'}(g_i)$, otherwise $x^{u_i} \notin \mathrm{in}_v(\mathrm{in}_{w'}(I)) = \mathrm{in}_{\tilde{w}}(I)$ (monomial ideal!)
(terms $\neq x^{u_i}$ in g_i live outside $\mathrm{in}_{\tilde{w}} I$)

$\Rightarrow \mathrm{in}_{w'}(g_i) - \mathrm{in}_w(g_i) \in \mathrm{in}_w(I)$ but contains no monomial in $\mathrm{in}_{\tilde{w}} I$. <u>Contr!</u>
$\mathrm{in}_{w'}(I) = \mathrm{in}_w(I)$ $\qquad$ $= \mathrm{in}_v(\mathrm{in}_w I)$ □

• $\overline{C_I[w]} = \{ z \in \overline{C_I[\tilde{w}]} : u_i \cdot z = -\mathrm{val}(c_{i,v}) + v \cdot z \ \ \forall_i\, \forall v \in \mathrm{Supp}(\mathrm{in}_w(g_i)) \}$ □
($>$ otherwise)

<u>Rmk</u> Clearly it is a face of $C_I[\tilde{w}]$ & defining hyperplanes are obtained from $\mathrm{supp}(\mathrm{in}_w(g_i))$ where g_i is the GB for $\tilde{w}$ constructed as in Step 1. ⇝ hyperplanes have an interpretation via initial ideals

• Lineality space $\supseteq \mathbb{R}\mathbb{1}$ for all cones ⇝ take quotient by it won't change the combinatorics & we drop dimension by 1

<u>Thm</u>: $\Sigma \underset{(\mathbb{L})}{=} \{\overline{C_I[w]}\}_{w \in \mathbb{R}^{n+1}}$ form a Γ_{val}-rational polyhedral complex supported on $\mathbb{R}^{n+1}/\mathbb{R}\cdot\mathbb{1} \simeq \mathbb{R}^n$ (uncanonically!). Call it the <u>Gröbner complex</u> of I

Q How many cells? <u>A</u> finitely many!

Lemma 2: $I \subset R$ homog. Then, there are only finitely many distinct monomial initial ideals $in_w(I)$.

Proof: By [Maclagan 2001] ("Antichains of monomial ideals are finite"), if they are not fin many monomial ideals $in_w(I)$, we can find $w_1, w_2 \in \mathbb{R}^{n+1}$ s.t $in_{w_2}(I) \subsetneq in_{w_1}(I)$ & both monomial ideals.

For $x^u \in in_{w_1}(I) \setminus in_{w_2}(I)$: By Lemma 2 L1X: monomials outside $in_{w_1}(I)$ are a K-basis of R/I, so $\exists f_u \in I$: $f_u = x^u + \sum_{\substack{x^v \notin in_{w_1}(I) \\ \text{finite}}} c_v x^v$.

- $in_{w_2}(f_u) \in in_{w_1}(I)$
- $in_{w_1}(I)$ monomial, so all terms of $\underbrace{in_{w_2}(f_u)}_{x^u \notin}$ lie in $in_{w_1}(I)$. Contra! □

  * only $x^u \in in_{w_1}(I)$.

Lemma 3: Maximal dimensional cells in Σ correspond to monomial ideals.

Proof idea: Find a polynomial g such that $\overline{C_I[w]}$ = chamber in the complement of $\mathcal{T}(V(g))$
& conversely: $w \in$ chamber $\Rightarrow \overline{C_I[w]}$ = chamber.
If $in_w I$ is not a monomial ideal, by Prop 1 we find $\tilde{w} \in \mathbb{R}^{n+1}$ with $\overline{C_I[w]} \subsetneq \overline{C_I[\tilde{w}]}$ so $\overline{C_I[w]}$ was not maxl dim'l.

Note The construction of g is explicit! See [MS p77]

- Use Gröbner Complex $\Sigma(I)$ to define a Universal GB $= \{g_1 \cdots g_s\} \subseteq I$ s.t $in_w I = \langle in_w(g_1), \dots, in_w(g_s)\rangle \ \forall w$.
- Next Time: From homogeneous ideals to Laurent polynomials via tropical bases.