Lecture XI: Tropical bases & tropical varieties

TODAY: Go from Homogeneous ideals in $R = K[x_0^{\pm}, \dots, x_n^{\pm}]$ to Laurent polynomials & study $\mathrm{in}_\omega I$ (nice set of generators: tropical bases) & $\mathcal{T}(I)$

Def: Given $I \subseteq K[x_1^{\pm}, \dots, x_n^{\pm}] = R$, and $\omega \in \mathbb{R}^n$, we define the initial ideal

$$\mathrm{in}_\omega(I) = \langle \mathrm{in}_\omega(f) : f \in R \rangle \subseteq K[x_1^{\pm}, \dots, x_n^{\pm}].$$

Recall: • in the homogeneous setting: for generic ω, $\mathrm{in}_\omega(I)$ is a monomial ideal

• In the Laurent setting, if $\mathrm{in}_\omega(I)$ contains a monomial $\Rightarrow \mathrm{in}_\omega I = \langle 1 \rangle$ (ω generic)

We are interested precisely in the non-generic case!

Q: How are $\mathrm{in}_\omega(I)$'s related in both settings? Idea: $I \subset R$ represents a subvariety of $(K^*)^n$. The homogeneous ideals represent subvarieties of $\mathbb{P}^n$.

$$X = V(I) \subset \underset{\displaystyle (K^*)^{n+1}/K^*}{\overset{\shortparallel}{(K^*)^n}} \subset \mathbb{P}^n \quad \rightsquigarrow \quad \overline{X} \subset \mathbb{P}^n \text{ (projective closure of } X\text{)}$$

is defined by I_{proj} (homog ideal!)

Def: $\tilde{f}_{(x_0, x_1, \dots, x_n)} = x_0^m f\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right)$ $\quad m = \text{degree of } f := \max\limits_{u \in \mathrm{supp}(f)} |u|$

is the homogenization of f wrt x_0.

Eg: $f = x^2 + xy + 1 \rightsquigarrow \tilde{f}_{(x,y,\underline{z})} = x^2 + xy + z^2 = z^2\left(\left(\frac{x}{z}\right)^2 + \frac{x}{z}\frac{y}{z} + 1\right)$

Set $S = K[x_1, \dots, x_n]$ $\rightarrow$ all exp in $\mathbb{N}_0^n$.

Def: $I_{proj} = \langle \tilde{f} : f \in I \cap S \rangle \subseteq K[x_0, \dots, x_n]$. $\rightsquigarrow I = (I_{proj} \cap \{x_0 = 1\}) \otimes_{K[x_1, \dots, x_n]} R$

• Would like to use $\Sigma(I_{proj}) \subseteq \mathbb{R}^{n+1}/\mathbb{R}\cdot\mathbb{1}$ $\simeq \underset{x_0}{\{0\}} \times \mathbb{R}^n$ [Gröbner complex] to study the various $\mathrm{in}_\omega I$ for ω in $\mathbb{R}^n$.

Prop: $I \subset R$, $\omega \in \mathbb{R}^n$. Then $\mathrm{in}_\omega(I)$ is the image of $\mathrm{in}_{(0,\omega)}(I_{proj})$ in $K[x_1^{\pm}, \dots, x_n^{\pm}]$ obtained by setting $x_0 = 1$.

Furthermore, every element of $\mathrm{in}_\omega I$ has the form $x^u g$ for $g = f(1, x_1, \dots, x_n)$ with $f \in \mathrm{in}_{(0,\omega)}(I_{proj})$ [$\underline{u}$ has the purpose of adjusting exponents from $\mathbb{N}^n$ to $\mathbb{Z}^n$.]

Proof: Write $f = \sum c_u x^u \in I \cap K[x_1, \dots, x_n] \rightsquigarrow \tilde{f} = \sum\limits_u c_u x^u x_0^{j_u}$ where $j_u = \max\limits_{c_v \neq 0} |v| - |u|$.

Then $\mathrm{trop}(f)(\omega) = \mathrm{trop}(\tilde{f})(0,\omega)$ & $\boxed{\mathrm{in}_{(0,\omega)} \tilde{f}\big|_{x_0=1} = \mathrm{in}_\omega(f)}$

Easy to see $\mathrm{in}_\omega(I) = \langle \mathrm{in}_\omega(f_1), \ldots, \mathrm{in}_\omega(f_s)\rangle$ for $f_i \in I \cap K[x_1, \ldots, x_n]$ (Noetherianness)

Then $\mathrm{in}_\omega(I) \subseteq \mathrm{in}_{(0,\omega)}(I_{proj})|_{x_0=1}$

Conversely: if $g \in I_{proj}$, then $g = x_0^j \tilde{f}$ for some j & $f_{(\underline{x})} = g(1, \underline{x})$

Choose a homog GB for I_{proj} with respect to $(0,\omega) \Rightarrow$ We show $\mathrm{in}_{(0,\omega)}(I_{proj})|_{t_0=1} \subseteq \mathrm{in}_\omega I$ $\square$

- Properties that translate from I_{proj} to I:

Lemma: $I \subset K[x_1^\pm, \ldots, x_n^\pm]$, $w \in \mathbb{R}^n$ (0) $\mathrm{in}_v(\mathrm{in}_\omega I)$ is v-homogeneous.

(1) $\mathrm{in}_v(\mathrm{in}_\omega I) = \mathrm{in}_\omega I$ for some $\underline{v} \in \mathbb{R}^n$, then $\mathrm{in}_\omega I$ is homogeneous wrt $\deg(x_i) = v_i$

(2) $f, g \in R$: $\mathrm{in}_\omega(fg) = \mathrm{in}_\omega(f)\, \mathrm{in}_\omega(g)$.

Proof (0) $\mathrm{in}_v(\mathrm{in}_\omega I) = \langle \mathrm{in}_v(g) : g \in \mathrm{in}_\omega I\rangle$

Given $g = \sum a_u x^u \in \mathbb{K}[x_1^\pm, \ldots, x_n^\pm]$, $\mathrm{in}_v(g) = \sum'_{u \cdot v = W} a_u x^u$ is v-homogeneous of degree $W = \mathrm{trop}(\mathrm{in}_v g)(v)$ (using trivial val on $\mathbb{K}$)

So $\mathrm{in}_v(\mathrm{in}_\omega I)$ is generated by v-homogeneous elements

(1) Follows from (0) if $\mathrm{in}_v(\mathrm{in}_\omega I) = \mathrm{in}_\omega I$.

§ Tropical bases $I \subset K[x_1^\pm, \ldots, x_n^\pm]$

- If (K, val) does not come with a splitting:

Def $\mathcal{T} = \{f_1, \ldots, f_s\}$ generators of I are a tropical basis if $\forall \omega \in \mathbb{R}^n$ $\exists f \in I$ where $\mathrm{trop}(f)_{(\omega)}$ is achieved only once $\iff$ $\exists f \in \mathcal{T}$ where $\mathrm{trop}(f)_{(\omega)}$ is achieved only once

IDEA: The finite set $\mathcal{T}$ witnesses $\mathrm{in}_\omega I$ containing a monomial. (for this we need a splitting!)

- If (K, val) has a splitting: $\mathcal{T}$ is a tropical basis if $\forall \omega \in \mathbb{R}^n$ $\mathrm{in}_\omega(I) = \langle 1 \rangle \iff \mathrm{in}_\omega(\mathcal{T}) = \{\mathrm{in}_\omega(f_1), \ldots, \mathrm{in}_\omega(f_s)\}$ contains a monomial.

Q: Do these exist? How to compute them?

Ex: If $f \in K[x_1^\pm, \ldots, x_n^\pm]$ then f is a tropical basis for $I = \langle f\rangle$.

Theorem: Every $I \subset K[x_1^\pm, \ldots, x_n^\pm]$ has a finite tropical basis

Proof idea: (1) Prove it when (K, val) has a splitting by going from I to I_{proj} & taking GB's for each all $\sigma \preceq \Sigma(I_{proj})$ (finitely many cells!), take $(0,\omega) \in \sigma^{\text{rel int}}$

We need to find witnesses for monomials in $\operatorname{in}_{(0,w)}(I_{proj})$ (even if the ideal is not monomial, it may contain one!)

If $x^u \in \operatorname{in}_{(0,w)}(I_{proj})$, pick $v \in \mathbb{R}^{n+1}$ generic & $\epsilon \ll 1$ with $\tilde{w} = w + \epsilon v$

$\operatorname{in}_v(\operatorname{in}_w I_{proj}) = \operatorname{in}_{\tilde{w}}(I_{proj})$ is a monomial ideal

Usual trick (Lecture X): Pick $f \in I_{proj}$ where $f = x^u + \sum_{\substack{|v|=d \\ x^v \notin \operatorname{in}_{\tilde{w}}(I_{proj})}} c_v x^v$ (deg d monomials outside $\operatorname{in}_{\tilde{w}} I_{proj}$ are a K-basis!)

wlog $\deg(u) = d$

By construction, if $w' \in C_{I_{proj}}[w]$ ($\operatorname{in}_{w'} I_{proj} = \operatorname{in}_w I_{proj}$), then

$\operatorname{in}_{(0,w')}(f) = x^u$ because $\operatorname{in}_{(0,w')}(f) - x^u \in \operatorname{in}_{(0,w)}(I_{proj})$

- If another term from f appears in $\operatorname{in}_{(0,w')}(f)$, then:
  - $\operatorname{in}_v(\operatorname{in}_{(0,w')}(f) - x^u) \notin \operatorname{in}_v(\operatorname{in}_{(0,w)} I_{proj}) = \operatorname{in}_{\tilde{w}} I_{proj}$
- If x^u does not appear in $\operatorname{in}_{(0,w')}(f)$, then:
  - $\operatorname{in}_v(\operatorname{in}_{(0,w')}(f)) \notin \operatorname{in}_v(\operatorname{in}_{(0,w)} I_{proj}) = \operatorname{in}_{\tilde{w}} I_{proj}$

We set $f'_\sigma = f' = f|_{x_0=1}$ & conclude $\operatorname{in}_{w'}(f')$ is a monomial.

Then $\mathcal{T} := \operatorname{gens}(I) \cup \{f'_\sigma : \sigma \in \Sigma(I_{proj})\}$ satisfies the defining properties of a trop basis.

(2) If K does not have a splitting, we do a base change to a valued field extn $L|K$ where $(L, \operatorname{val}_L)$ splits [Eg: $\mathbb{C}\{\{t\}\}|\mathbb{C}$, $K((\mathbb{R}))|K$ for K arbitrary]

$I_L = I L[x_1^\pm, \ldots, x_n^\pm]$. We conclude the result from the lemma below

Lemma: If a tropical basis for I_L exists, then there is one in $K[x_1^\pm, \ldots, x_n^\pm]$.

Pf/ Modify the trop basis in I_L to one in I by working with $\operatorname{supp}(f)$ $f \in \mathcal{T}$ & changing the coefficients so that:

$$\begin{cases} \cdot\ f' \text{ witness units in } \operatorname{in}_w I \\ \cdot\ f' \in K[x_1^\pm, \ldots, x_n^\pm] \\ \cdot\ f' \in I. \end{cases}$$

□

Warning: Tropical basis $\neq$ universal Gröbner basis

$\hookrightarrow \{g_1, \ldots, g_s\} \subseteq I$ s.t. $\operatorname{in}_w I = \langle \operatorname{in}_w(g_1), \ldots, \operatorname{in}_w(g_s)\rangle$ $\forall w$.

§2 Automorphisms of Tori: Monomial map $\phi^*: K[x_1^\pm, \ldots, x_m^\pm] \longrightarrow K[z_1^\pm, \ldots, z_n^\pm]$ induced by an matrix $A \in \mathbb{Z}^{n\times m} \leftrightarrow \mathbb{Z}^m \longrightarrow \mathbb{Z}^n$ $\phi^*(e_i) = u_i$ if $\phi^*(x_i) = z^{u_i}$

$A^T = \operatorname{trop}(\phi^*): \operatorname{Hom}(\mathbb{Z}^n, \mathbb{Z}) \longrightarrow \operatorname{Hom}(\mathbb{Z}^m, \mathbb{Z})$

$\quad\quad \cong \mathbb{Z}^n \quad\quad\quad\quad \cong \mathbb{Z}^m$

columns(A) = exponents $u_i \in \mathbb{Z}^n$ from $\phi^*(x_i) = z^{u_i}$

Note: $\mathrm{Trop}(\phi^*) : \Gamma_{val}^n \longrightarrow \Gamma_{val}^m$

because $val(\phi^*(y)) = (val(y^{a_1}), \ldots, val(y^{a_m})) = (a_1^T \cdot val(y), \ldots \cdot a_n^T val(y))$
$= A^T \cdot val(y) = \mathrm{trop}(\phi^*)(val(y))$

Lemma: $\phi^* : K[x_1^\pm, \ldots, x_m^\pm] \longrightarrow K[x_1^\pm, \ldots, x_n^\pm]$ monomial map
$I \subset K[x_1^\pm, \ldots, x_n^\pm]$ ideal, $I' = (\phi^*)^{-1}(I)$. Then

$$\phi^*(in_{\mathrm{trop}(\phi^*)(\omega)}(I')) \subseteq in_\omega I \qquad \forall \omega \in \mathbb{R}^n$$

So if we have ~~no~~ monomial in (RHS), we have no monomial in $in_{\mathrm{trop}(\phi)(\omega)}(I')$.

Coro: If ϕ^* is an automorphism of $(K^*)^n$, then $in_\omega I = \langle 1 \rangle \iff in_{\mathrm{trop}(\phi^*)(\omega)} I' = \langle 1 \rangle$.

Proofs: Exercises.

§3 Tropical Varieties:

Def: $I \subset K[x_1^\pm, \ldots, x_n^\pm]$, $X = V(I) \subseteq (K^*)^n$ The tropicalization $\mathrm{trop}(X)$ is

$$\mathrm{Trop}(X) = \bigcap_{f \in I} \mathcal{T}(V(f)) \subseteq \mathbb{R}^n \qquad (*)$$

Note: $\mathrm{Trop}(X)$ depends only on $\sqrt{I}$.

- Existence of a finite tropical basis for $I \Rightarrow$ the intersection in $(*)$ is finite: enough to run f over the tropical bases.

FUNDAMENTAL THM OF TROPICAL GEOMETRY: $K = \overline{K}$ with a nontrivial valuation

$I \subset K[x_1^\pm, \ldots, x_n^\pm] \rightsquigarrow X = V(I) \subseteq (K^*)^n$. The following sets agree:

(1) The tropical variety $\mathrm{Trop}(X)$ from $(*)$

(2) $\{\omega \in \mathbb{R}^n : in_\omega I \neq \langle 1 \rangle\}$

(3) closure of $\{(-val(y_1), \ldots, -val(y_n)) : y \in V(I)\} \subseteq \mathbb{R}^n$ (w/ Euclidean top)

Key: If $L|K$ is a valued field extension, then $\mathrm{Trop}(X) = \mathrm{Trop}(X_L)$. (from the trop basis construction)
so $\overline{K} = K$ with nontrivial valuation is not serious.