Lecture XII: Basics on matroids I

Upshot: Matroids generalize the notion of linear independence among a collection of vectors in a vector space over a field K. **Reference:** OXLEY'S BOOK.

Key example: full rank matrix $A = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 3 & 5 & 0 \end{bmatrix}$

Labels ① ② ③ ④ ⑤

Which subsets of $\{1,2,3,4,5\}$ give (i) linearly independent vectors?
(ii) maximally linearly indep. vectors?

Eg: $\{1,2\}$, $\{2\}$ are l.i., $\{2,4\}$, $\{5\}$ are NOT l.i.
Size of maximally l.i. $= 2$ (basis of Range(A).)

$\leadsto$ Aximatize / generalize these 2 notions.

Definition 1 Fix a finite set E = ground set (e.g. column labels).
A matroid M is a collection $\mathcal{B} = \mathcal{B}_{(M)}$ of subsets of E s.t.

(1) $\mathcal{B} \neq \emptyset$

(2) $B_1, B_2 \in \mathcal{B}$, $i \in B_1 \setminus B_2$, then $\exists\, j \in B_2 \setminus B_1$ st. $(B_1 \setminus \{i\}) \cup \{j\} \in \mathcal{B}$

[BASES EXCHANGE AXIOM]

$\mathcal{B}(M)$ = collection of bases of the matroid

Definition 2: E as in Def 1. A matroid M is a collection $\mathcal{I} = \mathcal{I}(M)$ of subsets of E s.t.

(1) $\mathcal{I} \ni \emptyset$

(2) $X, Y \in \mathcal{I}$ and $|Y| > |X| \Rightarrow \exists\, y \in Y \setminus X$ st $X \cup \{y\} \in \mathcal{I}$.

(3) $Y \in \mathcal{I}$, $X \subset Y \Rightarrow X \in \mathcal{I}$.

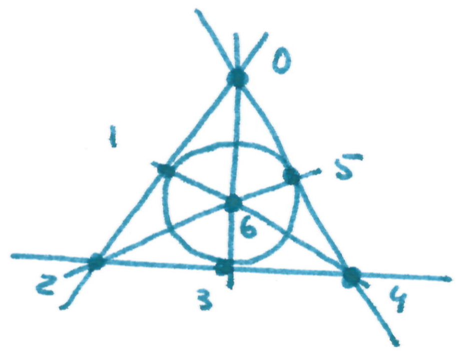
$\mathcal{I}(M)$ = collection of independent sets of the matroid.

• Remark: These 2 notions are equivalent: $\mathcal{B}(M) = \{X \in \mathcal{I}(M) : X \text{ mxl}\}$
$\mathcal{I}(M) = \{X \subseteq E \mid X \subseteq Y \text{ for some } Y \in \mathcal{B}(M)\}$

An example of a matroid can be constructed from the l.i. vectors of a full rank matrix $/K$ (as in the example). Matroids arising in this way are realizable over K
$\mathcal{B}(M) = \{$ maximally l.i. cols of $A\}$ $E = \{1, \ldots, n\}$ if $A \in K^{m \times n}$.

Remark: Not every matroid is realizable, & realizability depends on the field $\mathbb{F}$.

Example Fano matroid M



Take $E = \{0,1,2,3,4,5,6\}$

$\mathcal{B}(M) = \{ B \subset E : |B| = 3$ The pts in B are not on a common line or circle $\}$

Eg $\{013\} \in \mathcal{B}(M)$ but $\{012\} \notin \mathcal{B}(M)$.

- The Fano matroid is realizable over $\mathbb{F}_2$ but not over other fields

Why? Represents 7 sets of non-zero vectors in $\mathbb{F}_2^3$

lines / circles in the picture = lines in the projective plane over $\mathbb{F}_2$.

§ 2. Two alternative definitions

Definition 3: Fix a finite set E (ground set).

A matroid M is a collection $\mathcal{C} = \mathcal{C}(M)$ of subsets of E s.t.

(1) $\emptyset \notin \mathcal{C}$

(2) $C_1, C_2 \in \mathcal{C}$ & $C_1 \subseteq C_2 \Rightarrow C_1 = C_2$

(3) $C_1, C_2 \in \mathcal{C}$, $C_1 \neq C_2$ & $e \in C_1 \cap C_2$, then $\exists\, C_3 \in \mathcal{C}$ s.t.
$C_3 \subseteq (C_1 \cup C_2) - \{e\}$
[CIRCUIT ELIMINATION AXIOM]

$\mathcal{C}(M)$ = collection of circuits of the matroid.

Prop: Circuits = minimally dependent sets (so removing any element makes them l.i.)

Remark: To go from Definition 3 to Definition 2

$\mathcal{I}(M) = \{ I \subset E \mid I \not\supseteq C \text{ for all } C \in \mathcal{C}(M) \}$

- For the last definition, we'll need the notion of a rank function

In the realizable case : $rk(\{i_1, \dots, i_s\}) = rk(\mathrm{Span}(v_{i_1}, \dots, v_{i_s}))$

Modelled on this, we define :

Definition: Given E a finite set, a **rank function** on E is a function: $r : 2^E \longrightarrow \mathbb{Z}$ s.t

(1) $r(S) \geq 0$ & $r(S) \leq |S|$ for all $S \subset E$.

(2) $S \subseteq T \Rightarrow r(S) \leq r(T)$.

(3) $r(S \cup T) + r(S \cap T) \leq r(S) + r(T)$ [SUBMODULAR PROPERTY]

Why $\leq$? If V_S = span of vectors in S (in the realizable case), then $\dim(V_{S\cap T}) \leq \dim(V_S \cap V_T)$.

Definition 4: A rank function on E defines a matroid of rank $= r(E)$.

Remark: A set $I \subset E$ is independent if and only if $r(I) = |I|$.

(**Note**: $0 \leq r(\emptyset) \leq |\emptyset| = 0$ so $r(\emptyset) = 0$.)

Conversely, we can use $\mathcal{I}(M)$ to define the rank as $r(X) = |\text{largest independent set of } X|$

§3 Examples:

Ex 1: Uniform matroid of rank k on $E = \{1, \dots, n\}$. $= U(k,n)$

Bases = all k-element subsets of E.

$r(S) = \min\{|S|, k\}$ $\forall S \subseteq E$.

Ex 2: Graphical Matroids

Γ = a connected finite graph.

E = set of edges of Γ

- Bases = spanning trees (subtree in Γ containing all vertices of Γ)

(Eg  $\Gamma = K_4$ $E = \{1, \dots, 6\}$ Bases $= \{125, 154, 364, 263, 123, 234, 341, 412, 563, 254\}$)

- Rank function: $X \subset E$ $r(X) = \#\text{vertices} - \#\text{connected components}$

Eg: $r(123) = 4 - 1 = 3$ $r(\,|\;|\,) = 4 - 2 = 2$.

$r($$) = 4 - 1 = 3$

§4 Last definition: via Flats

In the realizable case: flats are associated with subspaces spanned by a collection of vectors in E & records all vectors from E in the given subspace

<u>Definition 5</u>: A set $S \subset E$ is a <u>flat</u> if $\forall j \in E \setminus S : r(S \cup \{j\}) > r(S)$

They can be viewed as taking <u>closure</u> of subsets in E. | <u>Rmk</u> E is always a flat

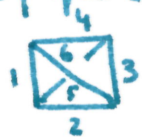
$$cl(X) = \{ e \in E : r(X \cup e) = r(X) \}$$

<u>Eg</u>: graphical matroids from $\Gamma = K_4$ $\quad E =$ edges

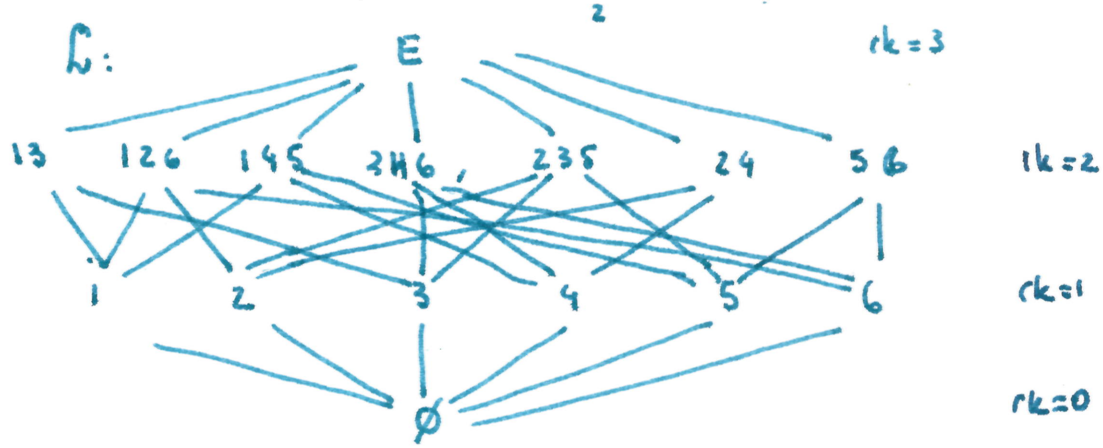
Proper Flats:
- $\{$ triangles in $\Gamma \}$
- $\{ i-j,\ k-\ell$, where $\{i,j,k,\ell\} = \{1,\dots,4\} \}$ $\Big\}$ rk=2
- $\{ i-j : i \neq j \}$ → rk=1

• Lattice of flats = poset of flats ordered by inclusion.

<u>Example</u>: $M(K_4)$



$\mathcal{L}$:



Next Time: Why is this poset a lattice?