Lecture XIII: Basics on Matroids II - Grassmannians

Recall (last time):

Def The lattice of flats of a matroid M is a poset under inclusion of flats. $(E, \mathcal{B}(M))$

(flats: $X \subset E$: $rk(X \cup \{x\}) > rk(X) \quad \forall x \notin X$).

§1 More on matroids

- Lattice structure on a poset $\mathcal{P}$
  - given $a, b \in \mathcal{P}$ we have a unique least upper bound $a \vee b$, called join
  - ——— we have a unique greatest lower bound $a \wedge b$, called meet.

Remark: Both operations $\wedge$ & $\vee$ are commutative & associative.

Every finite lattice has a $\hat{0}$ & $\hat{1}$, in our case $\hat{0} = \emptyset$, $\hat{1} = E$.

Def: A finite lattice $\mathcal{L}$ is called semimodular if

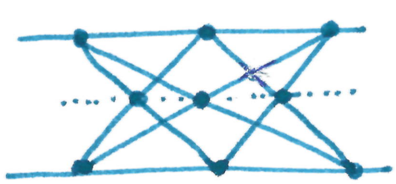
- given $a < b$ in $\mathcal{L}$, all maximal chains from a to b have the same length [Jordan-Dedekind chain cond.]
- $\forall x, y$ in $\mathcal{L}$: $h(x) + h(y) \geq h(x \vee y) + h(x \wedge y)$,

where $h : \mathcal{L} \longrightarrow \mathbb{Z}_{\geq 0}$ is the height function $h(x)$ = max'l length of a chain from $\hat{0}$ to x.

Def A geometric lattice is a finite semimodular lattice in which every element is a join of atoms (ie elements covering $\hat{0}$, ie of height = 1)

Theorem: A lattice $\mathcal{L}$ is geometric if and only if it is the lattice of flats of a matroid.

- Example of a non-realizable matroid: non-Pappus matroid M



NON-PAPPUS MATROID

Ground set = {9 pts}
dependencies determined by solid lines
Issue: M is not realizable over any field because the middle three vertices will be collinear (as the dashed line indicates). → adding this dependency gives the Pappus matroid.

Definition A loop $e \in E$ of a matroid is an element not contained in any independent set.

Equivalently: $r(\{e\}) = 0$.

- For the realizable case: $e = \{0\}$.

Definition: Given two non-loop elements $e, f \in E$, then e & f are parallel elements if $\{e,f\} \not\subset I$ for any independent set $I \in \mathcal{I}(M)$

Equivalently: $r(\{e\}) = r(\{f\}) = r(\{e,f\}) = 1$.

FACT: Given a matroid M of rank 2, we can turn M into a uniform matroid of rank 2 by deleting loops & identifying parallel elements.
$\leadsto$ [every pair of elements is a basis]

§2 The Grassmannian $Gr(d,n)$

Def: $Gr(d,n) = \{$ d-dim'l linear spaces in $K^n \}$ (K^n = fixed n-dim'l vector space / K)

$$\simeq M_{d\times n}(K)^{rk\, d} \big/ GL_d(K)$$ (rank d matrices of size $d\times n$ modulo row operations)

Plücker embedding: $\varphi : Gr(d,n) \hookrightarrow \mathbb{P}^{\binom{n}{d}-1}$

$$A \longmapsto [\, d\times d \text{ minors of } A\,] \quad \text{(homog. coords)}$$

Here: $\mathbb{P}^{\binom{n}{d}-1} = \Lambda^d K^n$

In coordinates: $\forall\, I \subset \{1,\dots,n\}$ d-element subset $I = \{i_1 < \dots < i_d\}$

we write $P_I(A) = \det\left(A_{\cdot}^{(i_1\cdots i_d)}\right) = \det\left(A^{(I)}\right)$

Eg: $d=2$, $n=5$ $\quad A = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 3 & 5 & 0 \end{bmatrix}$ $\quad p_{12}=1$, $p_{13}=3$, $p_{34}=10$, $p_{24}=0$, $p_{23}=-2$, $p_{14}=5$, $p_{i5}=0$ $\forall i$

The ideal defining the image of φ is the Plücker ideal $I_{d,n}$

- Generators of $I_{d,n}$ = Plücker quadratic relations.
  - (1) Keep the order of the indices in the subset I: $P_{i_1\cdots i_d} = \det\left((a_{j i_j})_{j=1}^{d}\right)$
  - (2) Quadratic "exchange" relations:

$$\sum_{l=1}^{d+1} P_{i_1\cdots i_{d-1}, j_l}\, P_{j_1\cdots \hat{j_l}\cdots j_{d+1}} = 0 \qquad \forall\, I = \{i_1\cdots i_{d-1}\},\ J = \{j_1\cdots j_{d+1}\}$$

Note
- $P_{I,i} = 0$ if $i \in I$, $|I| = d-1$.
- $P_{i_1\cdots i_j i_{j+1}\cdots i_d} := -P_{i_1\cdots i_{j+1} i_j \cdots i_d}$ $\leadsto$ use sign exchanges to put the indices in increasing order.

§3. The matroid stratification of $Gr(d,n)$

[Gelfand-Goresky-Macpherson-Serganova]

Def: $Gr_{\mathcal{J}}(d,n) = \{ x \in Gr(d,n) : p_I(x) = 0 \iff I \in \mathcal{J} \}$ ($\mathcal{J}$ induced by $Gr(d,n) \cap H$, H: coord plane in $\mathbb{P}(K^{\binom{n}{d}}/K^*)$)

Q: When is $Gr_{\mathcal{J}}(d,n) \neq \emptyset$?

Ans: $Gr_{\mathcal{J}}(d,n) \neq \emptyset \iff \mathcal{J}^c = \left(\binom{[n]}{d} \smallsetminus \mathcal{J}\right)$ is the set of bases of a rank d matroid on n realizable over K (The matrix realizing it will be an element in $Gr_{\mathcal{J}}(d,n)$)

- Strata of $Gr(d,n) \longleftrightarrow$ realizable matroids
- Issue 1: Stratification is not normal:

$$\overline{Gr_{\mathcal{J}}(d,n)} \neq \bigcup_{\mathcal{J}' \supseteq \mathcal{J}} Gr_{\mathcal{J}'}(d,n)$$

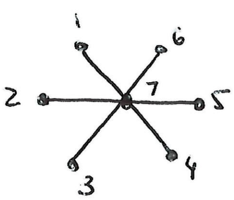
- Order: $\mathcal{J}' \supseteq \mathcal{J}$ is called the weak order on matroids ($M \leq M' \iff \mathcal{B}(M) \subseteq \mathcal{B}(M')$)
- [Sturmfels '87]: If $\overline{Gr_{\mathcal{J}}(d,n)} \cap Gr_{\mathcal{J}'}(d,n)$, then $\mathcal{J}' \leq \mathcal{J}$ (When char $K = 0$, $\overline{K} = K$)
- Converse does not hold:

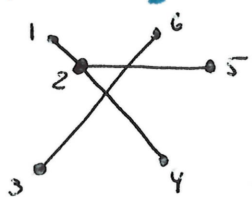
Example: $d=3$, $n=7$ over $K = \mathbb{C}$

$\mathcal{J} = \{147, 257, 367\}$

$\mathcal{J}' = \{ij7, 124\}_{i,j \neq 7}$

$\overline{Gr_{\mathcal{J}}(3,7)} \cap Gr_{\mathcal{J}'}(3,7) = \emptyset$


$= \mathcal{J}^c$


7 loop $= (\mathcal{J}')^c$

Issue 2:
- Each matroid stratum need not be irreducible nor reduced
- ______ can have arbitrarily bad singularities [Mnëv's Universality Thm]

- Example of rk 3 matroid M on 9 elements where $K[Gr_{\mathcal{B}(M)^c}(3,9)]$ is not Cohen-Macaulay

→ Solution: Positroid Stratification [Knutson-Lam-Speyer]

Thm [Corey-C]:
- The stratif. of $Gr(d,n)$ is normal for $d=2$ & $(d=3, n=6)$
- strata are reduced & irreducible in these cases.