Lecture XIV: The Tropical Grassmannian

<u>Recall</u>: $Gr(d,n) = \{d\text{-planes in } K^n\} = M_{d\times n}(K)_{\text{rk } d} \big/ GL_d(K) \overset{\varphi}{\hookrightarrow} \mathbb{P}^{\binom{n}{d}-1}$ Plücker embedding

$$X \longmapsto \Lambda^d X = \left(\det(X^{(I)})\right)_{I\in\binom{[n]}{d}}$$

Defined by the Plücker quadratic relations: $\left\{ \sum_{j\in J} \operatorname{sign}(j,I,J)\, P_{I\cup j}\, P_{J\smallsetminus j} \right\}$ $\quad |I|=d-1,\ |J|=d+1,\ |J\smallsetminus I|\geq 3$

$\rightsquigarrow I_{(d,n)}$ = Plücker ideal

Here $\operatorname{sign}(j,I,J) = (-1)^{\ell}$ where $\ell = \#\{k\in J : k>j\} + \#\{i\in I : i>j\}$.

- $\mathbb{P}^{\binom{n}{d}-1}$ is stratified by Tori, determined by vanishing of Plücker coordinates.

$\rightsquigarrow$ Matroid stratification [GGMS]:

$$Gr_{\mathcal{J}}(d,n) = \{ X \in Gr(d,n) \quad \varphi(X)_I = 0 \iff I\in\mathcal{J}\}$$

<u>Why?</u> $\mathcal{J}^c$ defines the bases of a realizable (over K) rank d matroid on n.

$$Gr_{\mathcal{J}}(d,n) \hookrightarrow (K^{\times})^{\mathcal{J}^c} \big/ K^{\times} \quad \text{with defining ideal } I_{\mathcal{J}^c} = \left(I_{d,n} + \langle P_\sigma : \sigma\in\mathcal{J}\rangle\right)^{+} \cap K[P_6^{\pm} : 6\in\mathcal{J}^c]$$

$\rightsquigarrow$ We tropicalize each strata & get a polyhedral fan in $\mathbb{R}^{\mathcal{J}^c}/\mathbb{R}\cdot\mathbb{1}$ (a subfan of the Gröbner fan of $I^{\text{proj}}_{\mathcal{J}^c}$)

§1 <u>TODAY</u>: We focus on $\mathcal{J}=\emptyset$ (uniform matroid $\mathcal{J}^c = U(d,n)$) $\rightsquigarrow Gr_{\emptyset}(d,n) \subseteq (K^{\times})^{\binom{n}{d}}/K^{\times}$.

We know
- $Gr_{\emptyset}(d,n)$ is irreducible
- $(K^{\times})^n \circlearrowright Gr_{\mathcal{J}}(d,n)$ by $(\underline{t} * P)_{i_1\ldots i_d} = t_{i_1}\cdots t_{i_d}\, P_{i_1\ldots i_d}$ (on each $\mathcal{J}$)

<u>Prop</u>: $\operatorname{Trop} Gr_{\emptyset}(d,n)$ is a pure $d(n-d)$-dimensional rational polyhedral fan in $\mathbb{R}^{\binom{n}{d}}/\mathbb{R}\cdot\mathbb{1}$ with an $(n-1)$-dim'l lineality space $\mathbb{L}/\mathbb{R}\cdot\mathbb{1}$ where

$$\mathbb{L} = \operatorname{span}\left\{ \sum_{i\in I} e_I \ : 1\leq i\leq n \right\} \subseteq \mathbb{R}^{\binom{n}{d}}$$

<u>Proof</u>. The dim & pureness statement follows from the Structure Thm of Trop varieties since $Gr_{\emptyset}(d,n)$ is irreducible of dimension $= d(n-d)$

- Each quadratic Plücker relation is homogeneous wrt vectors in $\mathbb{L}$, so each Gröbner cell $\left(\deg P_I = \sum_{i\in I} e_i \in \mathbb{Z}^n\right)$

$C_{I^{\text{proj}}_{U(d,n)}}[w] = C_{I^{\text{proj}}_{U(d,n)}}[w+\mathbb{L}]$ so $\mathbb{L} \subseteq$ lineality space of each cell. □

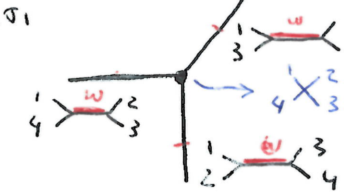
<u>Note</u> (1) $\mathbb{1}\in\mathbb{L}$ since sum(generators of $\mathbb{L}$) $= d\cdot\mathbb{1}$, (2) $\mathfrak{S}_n \circlearrowright \operatorname{Trop} Gr_{\emptyset}(d,n)$

<u>Example 1</u> $Gr(2,4)$ defined by $p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} \rightsquigarrow \mathrm{Trop}\, Gr_\phi(2,4)$ is the tropical hypersurface defined by $\max\{x_{12}+x_{34},\ x_{13}+x_{24},\ x_{14}+x_{23}\}$ in $\mathbb{R}^6/_{\mathbb{R}\cdot\mathbb{1}}$

$\mathbb{L} = \mathrm{span}\{e_{12}+e_{13}+e_{14},\ e_{12}+e_{23}+e_{24},\ e_{13}+e_{23}+e_{34},\ e_{14}+e_{24}+e_{34}\} = \begin{bmatrix} 1&1&1&0&0&0\\ 1&0&0&1&1&0\\ 0&1&0&1&0&1\\ 0&0&1&0&1&1 \end{bmatrix}$

3 ~~maximal~~ cones (unique non-winner):
- $\mathbb{L} + \mathbb{R}_{\geq 0}\langle \overset{12\ 13\ 14\ 23\ 24\ 34}{(0,1,1,1,1,0)}\rangle = \sigma_1$
- $\mathbb{L} + \mathbb{R}_{\geq 0}\langle(1\,0\,1\,1\,0\,1)\rangle$
- $\mathbb{L} + \mathbb{R}_{\geq 0}\langle(1\,1\,0\,0\,1\,1)\rangle$

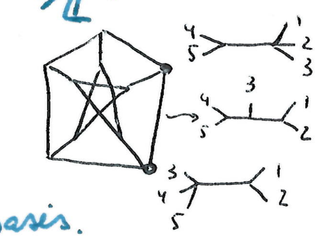
[$\dim = 4-3 = 1$ in $\mathbb{R}^6/_{\mathbb{L}}$]



$\mathbb{R}^6/_{\mathbb{L}} \simeq \mathbb{R}^2$ via projection to $(x_{12}, x_{13}, 0, 0, 0, 0)$

<u>Example 2</u> $Gr_\phi(2,5) \subseteq \mathbb{R}^{10}/_{\mathbb{R}\cdot\mathbb{1}}$ is a 6-dim'l fan, ~~and~~ 2-dim'l in $\mathbb{R}^{10}/_{\mathbb{L}}$.

f-vector $= (1, 10, 15)$ $\rightsquigarrow$ cone over Petersen graph
(1 = vertex, 10 = rays, 15 = 2-dim'l cones)



• <u>In general</u>: To compute $\mathrm{Trop}(Gr_\phi(d,n))$, we need to find a tropical basis.
The case for $d=2$ is particularly nice:

<u>Thm</u> [Speyer-Sturmfels] $\mathrm{Trop}\, Gr_\phi(2,n) \subseteq \mathbb{R}^{\binom{n}{2}}/_{\mathbb{L}}$ is the <u>space</u> of <u>phylogenetic trees</u> $\mathcal{T}_n$ on n leaves, labelled 1 through n (cf [Billera-Holmes-Vogtmann])

The quadratic Plücker eqns form a tropical basis.

<u>§2: The space of Phylogenetic Trees</u>

$(T, w) \in \mathcal{T}_n$ for
- T a graph with no cycles, no deg 2 vertices & n labelled leaves 1 through n
- $w: E(T) \longrightarrow \mathbb{R}$ (weight) & $w(e) \geq 0$ if $e \in E(T)$ <u>not</u> adjacent to any leaf.

• If T is trivalent, it has $2n-3$ edges, $n-3$ non-leaf edges

$\rightsquigarrow$ For each T we get $\mathbb{R}^n \times \mathbb{R}^{n-3}_{\geq 0}$. ($(2n-5)!! = (2n-5)(2n-7)\cdots 5\cdot 3\cdot 1$ of them)

• If $w(e) \to 0$ ($0 \leq$), we get a tree T' by contracting the edge e in T.

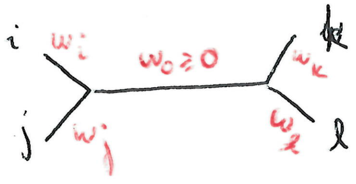
• Using contractions we glue the spaces $\mathbb{R}^n \times \mathbb{R}^{n-3}_{\geq 0}$ together.

$\to$ Fan structure: cones labelled by trees.

$\mathcal{C}_T \leq \mathcal{C}_{T'} \iff T$ is a coarsening of T'.

Lineality space $= n$ leaf edges.

• $(T, w) \rightsquigarrow \underline{d} \in \mathbb{R}^{\binom{n}{2}}$ <u>tree distance</u> as $d_{pq} = \sum_{e: p \to q} w(e)$ $(p \neq q)$



$$\begin{cases} d_{ij} = w_i + w_j \\ d_{ik} = w_i + w_0 + w_k \\ d_{i\ell} = w_i + w_0 + w_k \end{cases}, \text{ etc}$$

Note: $\max\{d_{ij} + d_{k\ell},\ \underline{d_{ik} + d_{j\ell}},\ \underline{d_{i\ell} + d_{jk}}\}$ is attained twice

$\quad \| \; w_i + w_j + w_k + w_k \qquad = w_i + w_j + w_k + w_\ell + 2w_0$ ≥ 0

$= \operatorname{trop}(p_{ij} \odot p_{k\ell} \oplus p_{ik} p_{j\ell} \oplus p_{i\ell} p_{jk})$

Note: non-max term gives the 2 cherries (ij), (k, ℓ) in the tree T

If max is attained 3 times, then $w_0 = 0$ & tree is 

= star tree on 4 leaves.