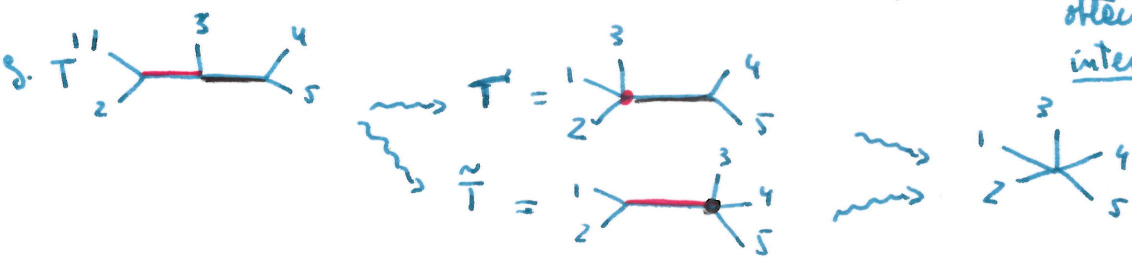
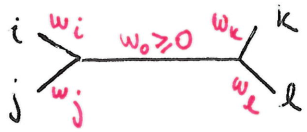
Lecture XV: The Tropical Grassmannian II.

Recall: The space $\mathcal{T}_n$ of Phylogenetic Trees:
- cones labelled by trees T
- $\mathcal{C}_T \subseteq \mathcal{C}_{T'} \iff T$ is a coarsening of T' obtained by contracting internal edges (not adjac. to a leaf)

E.g.  star tree

$w: E(T) \longrightarrow \mathbb{R}$ weight function, $w(e) \geq 0$ if $e \in E(T)$ internal edge

given $(T, w) \rightsquigarrow d \in \mathbb{R}^{\binom{n}{2}}$ tree distance $d_{ij} = \sum_{e: i \to j} w(e)$ (=0 if i=j)



quartet $(ij|kl)$

(*) $\max\{d_{ij}+d_{kl},\ \underline{d_{ik}+d_{jl}},\ \underline{d_{il}+d_{jk}}\} = \operatorname{trop}(p_{ij}p_{kl} - p_{ik}p_{jl} + p_{il}p_{jk})$

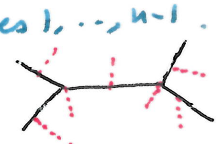
$= d_{ij} + d_{kl} + 2w_0 \geq 0$

non-max Term = 2 cherries.

Lemma (4 Pt condition) A pt $d \in \mathbb{R}^{\binom{n}{2}}$ is a tree distance (comes from (T,w)) $\iff$ for all i,j,k,l the max in (*) is attained twice.

Pf/ (1) Combinatorially, T is defined by its list of quartets (subtrees spanned by 4 leaves)

(2) We use (*) to decide what's the comb. type of each quartet: $(ij|kl)$, $(ik|jl)$, $(il|jk)$ or star tree.

(3) Proceed by induction on n & construct T' spanned by leaves $1, \ldots, n-1$. Use the conditions (*) involving $l = n$ to see where to attach leaf n 

(4) Weights are recovered by linear algebra (from T & d)

Note: (2) & (3) can be achieved by Neighbor-Joining Algorithm.

. Effect of weights on leaf edges: w_i contributes to entries d_{ij} for $j \neq i$

$\rightsquigarrow$ n vectors $l_i = \sum_{j \neq i} e_{ij}$ span the lineality space $\mathbb{L}$ in $\mathbb{R}^{\binom{n}{2}}$ & contain $\mathbb{1}$.

Pf of [Speyer-Sturmfels]: . By 4 pt condition $\operatorname{Trop} Gr_{\emptyset}(2,n) \subseteq \mathcal{T}_n$

For the converse, need to show that quadratic Plücker relations are a tropical basis. Enough to show: they are a Gröbner basis for any monomial term $<$ refining the weight order $<_w$ given w in $\operatorname{relint}(\mathcal{C}_T)$.

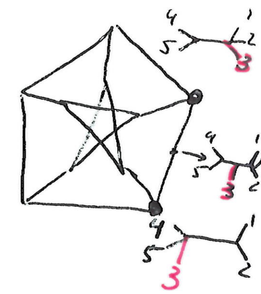
§1 Combinatorics of $\mathcal{T}_n$

• Max'l cones $\longleftrightarrow$ trivalent trees : $(2n-5)!! = (2n-5)(2n-7)\cdots 5\cdot 3$ of them.

Each $\mathcal{C}_T \simeq \mathbb{R}^n \times \mathbb{R}^{\ell}_{\geq 0}$ $\quad \ell = \#$ internal edges of T ($=2n-3$ if T trivalent)

Each non-leaf edge induces a split on the leaves of T : $[n] = I \sqcup I^c$, $\quad |I|, |I^c| \geq 2$

$\rightsquigarrow I \xrightarrow[wt=1]{e} I^c$ gives a split metric $e^{I,I^c} = \sum_{\substack{i\in I\\ j\in I^c}} e_{ij}$ in $\mathbb{R}^{\binom{n}{2}}$.



Prop: $\mathcal{C}_T = \mathbb{R}_{\geq 0}\langle e^{I,I^c} : e$ non leaf edge in T inducing split $I \sqcup I^c \rangle + \mathbb{R}^n$

$$\ni \; w = \sum_{\substack{e\in E(T)\\ \text{internal}}} w(e)\, e^{I,I^c} + \underbrace{\sum_{i=1}^{n} w_i \Big(\sum_{j\neq i} e_{ij}\Big)}_{\text{in lineality space of all } \mathcal{C}_T}$$

• Later in the course:
- These split trees correspond to divisors on $\overline{M}_{0,n}$ (Deligne-Mumford compactif.)
- $\overline{M}_{0,n}$ = toric variety with fan $\mathcal{T}_n / L$ in $\mathbb{R}^{\binom{n}{2}-n}$.
- [Cartwright-Maclagan] use other fans to get alternative compactifications of $M_{0,n}$.

Prop: $\{e^{I_j, I_j^c}\}_j$ are rays of a cone $\mathcal{C}_T \iff$ The splits are pairwise compatible ($(A,A^c), (B,B^c)$ are comp if one of 4 intersections is $\emptyset$). So $\mathcal{T}_n$ is a flag complex (minimal non-faces are pairs of vertices)

§2 Other Strata?

$Gr_{\mathcal{J}}(2,n) = \{X \in Gr(2,n) \;\; p(X)_I = 0 \iff I \in \mathcal{J}\}$ $\quad \mathcal{J}$ = pairs of l.d columns of $2\times n$ matrix X.

2 types of columns =
- zero columns (loops of $\mathcal{J}^c$)
- nonzero columns $\to$ group them as rank 1 flats = max'l l.d columns that are not loops.

After relabelling :
$$X = \Big(\overbrace{X_1}^{1\cdots r_1} \,\Big|\, \overbrace{X_2}^{r_1+1 \cdots r_2} \,\Big|\, \cdots \,\Big|\, X_m \,\Big|\, 0 \Big)$$
$X_1, X_2, \ldots, X_m$ l.d; X_1, X_2 l.i

$B_1 = \{1, \ldots, r_1\}$
$B_2 = \{r_1+1, \ldots, r_2\}$
$\vdots$
$\bigsqcup_{i=1}^{m} B_i = [n] \smallsetminus \text{loops}$

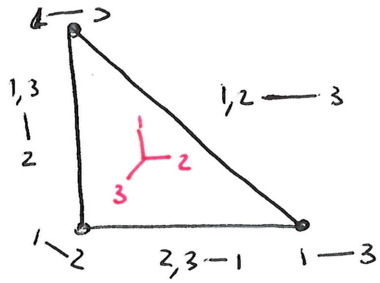
Project X to 1^{st} column of each X_i & get an matrix in $Gr_{\emptyset}(2,m)$

Thm [C.] $\operatorname{Trop} Gr_{\mathcal{J}}(2,n) = \mathcal{T}_m$ (labelled by 1^{st} element of each B_i) $\times \mathbb{R}^{n-\#\text{loops}-m}$

Better : label the leaves in $\mathcal{T}_m$ by the sets $B_1, \ldots, B_m$.

We glue these spaces together to get $\operatorname{Trop} Gr(2,n)$ (in $\mathbb{TP}^{\binom{n}{2}-1} = \overline{\mathbb{R}}^{\binom{n}{2}} / \mathbb{R}\cdot\mathbb{1}$) where $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$ (& usual topology). Call the resulting space $\overline{\mathcal{T}_n}$ the generalized space of phylogenetic trees.

Example 1



f-vector $\overline{\mathcal{J}_3} = (3,3,1)$

Example 2: f-vector $\overline{\mathcal{J}_4} = (6, 12, 11, 7, 3)$.

§3 Higher Grassmannians:

- Characteristic dependent: $Gr_\emptyset(3,7)$
- quadratic Plücker relns are not a tropical basis (unless $Gr(2,n)$, $Gr(3,6)$).
- Trop $Gr_\emptyset(3,6)$ is a 9-dim'l fan in $\mathbb{R}^{\binom{6}{3}}/\mathbb{R}\cdot\mathbb{1}$ with a unique coarsest fan structure
  - modulo $\mathbb{L}$, the f-vector is $(1, 65, 535, 1350, 1005)$ → 990 tetrahedra, 15 bipyramids 
    ↳ 7 ones modulo $\mathfrak{S}_6$-action
  - not a flag complex (inner triangles in the bipyramides are not faces)
    ↳ def: minimal faces are pairs of vertices
- Combinatorics of root systems appear ⇝ HARD!
- Rays of all Trop $Gr_\emptyset(3,n)$ are known [Herrmann-Joswig-Speyer], but that's about it for $n \geq 8$.

§4 Tropical Grassmannian as a tropical moduli space:

Recall $Gr(d,n) = \{d\text{-planes in } K^n\} = \{(d-1)\text{-planes in } \mathbb{P}^{n-1}\}$

Explicit bijection: Given $p \in Gr_\emptyset(d,n)$ & any $I = \{i_1, \ldots, i_{d+1}\} \in \binom{[n]}{d+1}$, we form $\ell_{I,p} = \sum_{j=1}^{d+1} (-1)^j \, p_{I \smallsetminus i_j} \, x_{i_j}$

The ideal $\mathcal{L}_p = \langle \ell_{I,p} : I \in \binom{[n]}{d+1} \rangle \subseteq K[x_1^\pm, \ldots, x_n^\pm]$ is a linear & defines $L \cap (K^\times)^n$ where L is a d-plane in K^n. $(L := V(\mathcal{L}_p \cap K[x_1, \ldots, x_n]))$

Converse: any d-plane in K^n = row space of a matrix $B \in K^{d\times n}$ of rank d
⇝ get $p \in Gr(d,n)$ as $\varphi(B) =: p$.

Note: If $p \in Gr_{\mathcal{J}}(d,n)$ with $\mathcal{J} \neq \emptyset$ & $\mathcal{J}^c$ has no loops, the same construction works. If $\mathcal{J}^c$ has loops, we know $\{x_i : i \text{ loop}\} \subseteq \mathcal{L}_p$ so we first project $\mathcal{L}_p$ to $K[x_j : j \text{ non loop}]$ & then insert the non-loops.

Key lemma: • The circuits of a linear ideal (= linear forms with minimal support) form a tropical basis.

• given $L \rightsquigarrow \{\ell_{I,p}\}_I$ are its circuits.
$(p \text{ in } Gr_\emptyset(d,n))$

Note: The circuits are not always a minimal basis (see HW2)

Main Thm: The bijection $Gr_\emptyset(d,n) \longrightarrow \{d\text{-planes in } K^n\}$ viewed in $(K^\times)^n$

$$p \longmapsto V(\mathcal{L}_p)$$

induces a bijection $\operatorname{Trop} Gr_\emptyset(d,n) \cap \Gamma_{val}^{\binom{n}{d}-1} \longrightarrow$ tropicalization of d-planes in K^n with matroid $U(d,n)$

$$w \longmapsto \bigcap_{I\in\binom{[n]}{d+1}} V\Big(\bigoplus_{i\in I} w_{I\smallsetminus i}\odot X_i\Big) =: L_w$$

Note: If val on K is trivial, not interesting!

Proof: given w, we find $p\in Gr_\emptyset(d,n)$ with $-\operatorname{val}(p_-) = w$.

Then $\operatorname{trop}(\ell_{I,p}) = \bigoplus_{i\in I} w_{I\smallsetminus i}\odot X_i$. & use Key Lemma.

• Surjective by construction

• injective: Recover w from $\operatorname{Trop}(L\cap (K^\times)^n)$ for L a d-plane in K^n with matroid $U(d,n)$.

Note w will be recovered up to $\mathbb{R}\cdot\underline{1}$. [ie find inverse map!]

Enough: For any $J\in\binom{[n]}{d-1}$ & $k,l\notin J$ want to recover $w_{J\cup\{l\}} - w_{J\cup\{k\}}$ directly from $\operatorname{Trop}(L\cap(K^\times)^n)$ [can get any $w_I - w_{I'}$ for $I, I' \in\binom{[n]}{d}$ from the pairs alone].

Pick $C \ll 0$ in Γ_{val} (Assume K has nontrivial valn) & $c\in K$ with $-\operatorname{val}(c) = C$.

• Fix J, $k,l\notin J$ as above.

Given L we can find a unique pt $x = (x_1,\ldots,x_n)$ in L with $x_k = 1$, $x_j = c$ $\forall j\in J$ (matroid of L is $U(d,n)$ so $\pi_{J\cup\{k\}}: L \twoheadrightarrow \mathbb{R}^d$) Pick ! $p\in Gr_\emptyset(d,n)$ with $\mathcal{L}_p = L$

Take circuit $I = J\cup\{k,l\}$ ($|I| = d+1$ & matroid is $U(d,n)$) then:

$$\ell_{I,p}: \; x_l\, p_{J\cup\{k\}} \pm p_{J\cup\{l\}} + \sum_{j\in J} \pm c\, p_{J\smallsetminus\{j\}\cup\{k,l\}} = 0$$

The point $u := -\operatorname{val}(x) \in L_w$ satisfies

• $u_k = 0$

• $u_j = C$ $\forall j\in J$

• $\operatorname{trop}(\ell_{I,p})(w) = \max\{\underline{u_l + w_{J\cup\{k\}}},\ \underline{w_{J\cup\{l\}}},\ C + w_{J\smallsetminus\{j\}\cup\{k,l\}}\}$

Given w, since $C\ll 0$, max is achieved at underlined terms & $w_{J\cup\{l\}} - w_{J\cup\{k\}} = u_l$.