Lecture XVI: Tropical linear spaces I

TODAY: Constant coefficient case (e.g. defined over $\mathbb{C}$)

§1. **GOAL**: If V is a linear subspace of $(K^*)^n$ (defined over $\mathbb{K}$) describe $\operatorname{Trop} V \subseteq \mathbb{R}^n$ $\left(\operatorname{Trop} V \subseteq \mathbb{R}^n / \mathbb{R}\cdot\mathbb{1} \right)$

Lemma: $w \in \operatorname{Trop} V \iff$ for each circuit of $I(V)$ $a_1 x_{i_1} + \cdots + a_k x_{i_k} = 0$ then $\max\{w_{i_1}, \ldots, w_{i_k}\}$ is achieved twice

Here: circuit = linear equation of $I(V)$ with minimal support. (circuits of the matroid of columns of $A = \begin{bmatrix} v_1 \\ \vdots \\ v_d \end{bmatrix}$ $\{v_1, \ldots, v_d\}$ basis of V.

Example: V = rowspace $\begin{vmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 \end{vmatrix}$ $(rk = 3)$

$I(V) = \langle x_1 - x_2 + x_3,\ x_4 - 2x_3 \rangle$ $\rightsquigarrow$ minimal dependencies of cols of A

Circuits: $\{123, 34, 124)$

By Lemma $\operatorname{Trop}(V) = \{\max(w_1, w_2, w_3), \max(w_1, w_2, w_4), \max(w_3, w_4)$ attained twice $\}$

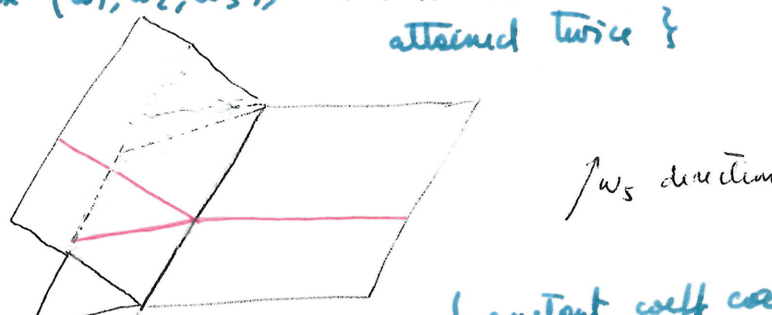
- w_5 can be anything

$\{1\} \subseteq \{1234\}$ • $w_1 < w_2 = w_3 = w_4$ $(-e_1)$

$\{2\} \subseteq \{1234\}$ • $w_2 < w_1 = w_3 = w_4$ $(-e_2)$

$\{34\} \subseteq \{1234\}$ • $w_3 = w_4 < w_1 = w_2$ $(-e_3 - e_4)$

(flags of flats in [4])



(constant coeff case!)

Obs: $\operatorname{Trop} V$ depends ONLY on the matroid of V combinatorics is determined by the lattice of flats. $\operatorname{Trop} V$ = Bergman fan of the matroid.

Recall:

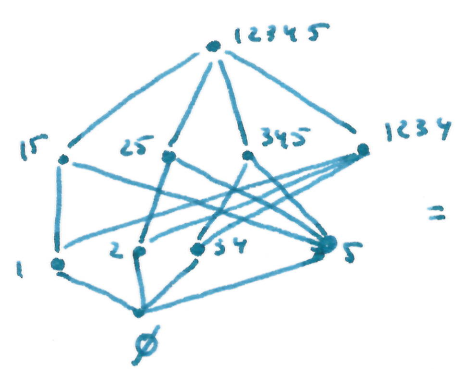
- Lattice of flats $\mathcal{L}$ = poset of flats ordered by inclusion.
- Flat = $S \subseteq [n]$ s.t. $\forall j \in [n] \setminus S$ $rk(S \cup \{j\}) > r(S)$

(For realizable matroids = subspaces in column space of the matrix A)

Example (cont.)

Flats:

- $\emptyset$
- $1, 2, 34, 5$ (lines)
- $15, 25, 345, 1234$ (planes)
- 12345 (3-space)

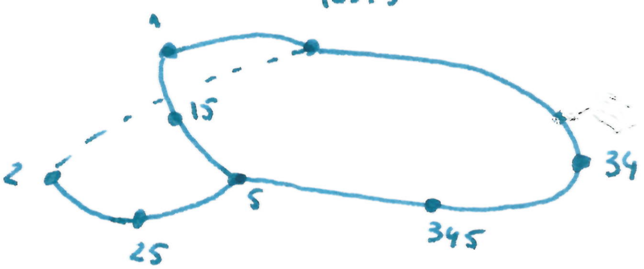


$= \mathcal{L}$

$\rightsquigarrow$ Order complex of posets $= \Delta(\mathcal{L} \setminus \{\hat{0}, \hat{1}\}$ $\hat{0}$ = min element $(= \emptyset$ for us) $\hat{1}$ = top " $(= [n]$ —)

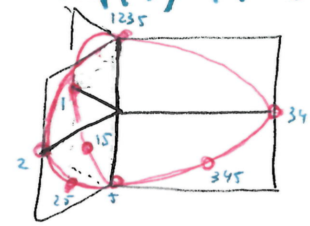
- vertices = elements of the Poset (flats)
- faces = chains in the Poset (flags of flats)

Example: (cont.)



$= \Delta(\mathcal{L} \setminus \{0, 1\})$

$= \mathrm{Trop}(V) \cap S^4$ with additional subdivisions



Main Thm [Ardila, Klivans]

$\mathrm{Trop}(V) \cap \text{unit sphere} \text{ "}\underset{\substack{\text{combinatorially} \\ \text{(up to subdivisions)}}}{=}\text{" } \Delta(\mathcal{L}_V \setminus \{\hat{0}, \hat{1}\})$

↪ . pure, shellable complex

. Topologically = wedge of $(\dim V - 2)$-dim'l spheres (Björner)

§3 Concrete embeddings in $\mathbb{R}^n$ via Bergman fan of a matroid

$V \rightsquigarrow$ rank d matroid M on $[n]$ (cols of A st rowspan$(A) = V$)

$\mathcal{B} = \mathcal{B}(M)$ bases of M

Given $w \in \mathbb{R}^n$ & $B = \{b_1, \dots, b_d\}$ in $\mathcal{B}$: weight$(B) = w_B := w_{b_1} + \cdots + w_{b_d}$

$M_w := \{B \in \mathcal{B} \mid w_B \text{ is MINIMAL}\}$ is the collection of bases of a matroid

Why? Characterization of matroids via matroid polytopes:

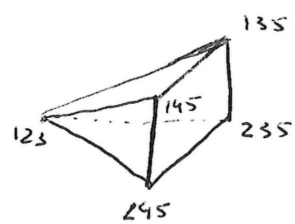
Def: Given $M \rightsquigarrow P_M = \mathrm{conv\,hull}(\text{incidence vectors } (v_B) : B \in \mathcal{B}(M)) \subseteq \mathbb{R}^n$
(= matroid polytope of M)

$$v_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{else} \end{cases}$$

Example (cont.) $\mathcal{B}(M) = \{125, 135, 145, 235, 245\}$ ($\subseteq \mathbb{R}^5$

$P_M = \mathrm{conv}(11001, 10101, 10011, 01101, 01011) \subseteq \mathbb{R}^5$

Note: $\mathrm{span}(P_M) = \begin{cases} x_1 + x_2 + x_3 + x_4 + x_5 = 3 = \mathrm{rk}(M) \\ x_5 = 1 \end{cases}$ $\rightsquigarrow$ 3-dim'l



Thm [Gelfand Goresky-McPherson-Serganova]

A 0-1 polytope P (ie conv hull of 0/1-vectors) is a matroid polytope

$\Leftrightarrow$ all edges of P are of the form $e_i - e_j$ (standard basis of $\mathbb{R}^n = \{e_1, \dots, e_n\}$)

Why? Neighboring vertices $\Leftrightarrow$ bases of M related by the exchange axiom.

• $M_w = \{B \mid v_B \text{ is a vertex in the face of } P_M \text{ dual to } -w\}$ ~~as~~ P_{M_w} satisfies [GGMS] criterion!

Def The Bergman fan of M is $\widetilde{\mathcal{B}}(M) = \{ w \in \mathbb{R}^n : M_w \text{ has no loops} \}$

Cells of $\widetilde{\mathcal{B}}(M) \longleftrightarrow$ chains of flats of M (cells of $\Delta(\mathcal{L} \setminus \{\hat{0}, \hat{1}\})$)

$\sigma = \{ F_1 \subsetneq \cdots \subsetneq F_m \} \rightsquigarrow e_{F_i} = -\sum_{j \in F_i} e_j \in \mathbb{R}^n$ for $i = 1, \dots, n$

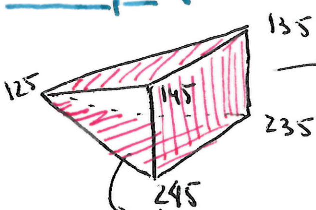
Cell associated to $\sigma = \mathbb{R} \cdot \underline{1} + \mathbb{R}_{\geq 0} \langle e_{F_1}, \dots, e_{F_m} \rangle$

↳ corresponds to $M_{\underline{1}} = \mathcal{B}(M)$ (all bases)

Prop [Sturmfels] $\mathrm{Trop}(V)$ is the fan dual to the loopless faces of P_M

(i.e. vertices v_B in the face give a matroid ($= M_w$) without loops)

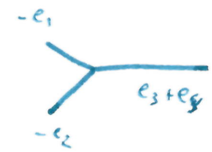
Example (cont.)



→ back: 4 is a loop.

front: 3 is loop. ($3 \notin 125, 145, 245$)

$\rightsquigarrow$ combinatorial =



$\times \; \mathbb{R} = \langle e_5 \rangle$

In general: k-dim'l faces of $\mathrm{Trop}\, V \longleftrightarrow (\dim P_M - k)$-dim'l loopless faces of P_M.

Next time: Arbitrary coefficient case.