Lecture XVII: Tropical linear spaces II

Recall: • Trop linear spaces $\mathrm{Trop}(V)$ over constant coefficients (I linear ideal defined over k) = order complex of poset of flats of the associated matroid M $\Delta(\mathcal{L} \setminus \{\hat{0}, \hat{1}\})$

same support

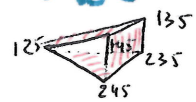
• We understand the topology & combinatorics: $\underset{\mathbb{R}\mathbb{1}}{\underline{\mathrm{Trop}(V)}} \cap S^{n-2} \simeq$ wedge of dim $\mathcal{L}$-2 dim'l spheres.

• Embedded fan = $\widetilde{\mathcal{B}}(M) = \{ w \in \mathbb{R}^n : M_w \text{ has no loops} \}$

$$M_w = \{ B \in \mathrm{Basis}(M) : w_B = \sum_{i \in B} w_i \text{ is minimal} \}$$

Prop [Sturmfels]: $\mathrm{Trop}(V)$ is the fan dual to the loopless faces of the matroid polytope $P_M = \text{convex hull} \{ v_B = \sum_{i \in B} e_i : B \in \mathrm{Basis}(M) \}$

§1. Trop Linear Spaces w/ arbitrary coefficients:



dual to $\succ\!\!-\ \times \mathbb{R}$ → basis of V

• OPTION 1: Tropical basis = circuits of the matroid (= cols of $\begin{bmatrix} v_1 \\ \vdots \\ v_d \end{bmatrix}$)

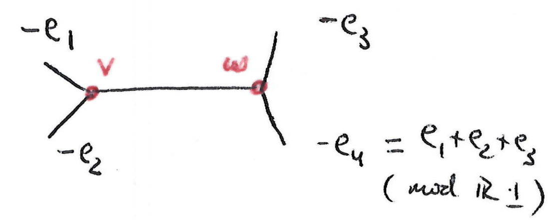
• OPTION 2: They are obtained by locally gluing tropical linear spaces with constant coefficients [Rincon: "Local tropical linear spaces"]

Key Prop ("Gluing lemma") Let V be a linear space in K^n meeting $(K^*)^n$ & let v be in Trop V. The local cone (or star) at v:

$$\mathrm{Star}_{\mathrm{Trop}\,V}(v) = \{ u \mid v + \varepsilon u \in \mathrm{Trop}\,V \text{ for } 0 < \varepsilon \ll 1 \}$$

is a tropical linear space with constant coefficients.

Example V = row space $\begin{array}{c} \begin{vmatrix} 1 & t & t^2 & t^3 \\ t^3 & t^2 & t & 1 \end{vmatrix} . \\ \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \end{array}$



Circuits: 123, 124, 134, 234 ($M = U(2,4)$)

$I(V) = \langle (t-t^3)x_1 - (1-t^4)x_2 + (t-t^3)x_3,\ (t-t^5)x_1 - (1-t^6)x_2 + (t^2-t^4)x_4,$
$(t^2-t^4)x_1 - (1-t^6)x_3 + (t-t^5)x_4,\ t x_2 - (1+t^2)x_3 + t x_4 \rangle$

3x3 minors

$\mathrm{Trop}\,V = \{ \underline{w} : \max\{-1+w_1, w_2, -1+w_3\},\ \max\{-1+w_1, w_2, -2+w_4\},$
$\max\{-2+w_1, w_3, -1+w_4\},\ \max\{-1+w_2, w_3, -1+w_4\}$ attained twice $\}$

• Equality in (1) ⇝ $v = (1, 0, 1, 2)$ ⇝ rays: $-e_1, -e_2, e_1+e_2$

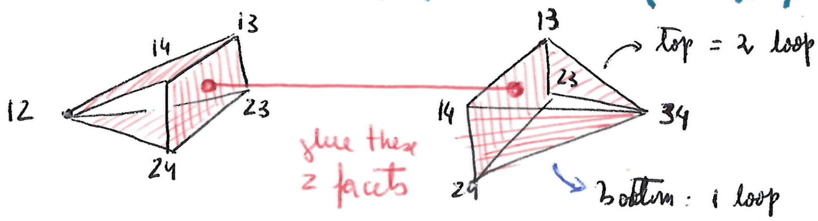
• Equality in (4) ⇝ $w = (1, 0, -1, 0) = (2, 1, 0, 1)$ ⇝ rays: $-e_3, -e_4, e_3+e_4$

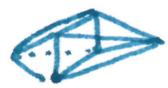
} opposite modulo $\mathbb{R}\mathbb{1}$.

Q: How do we glue the 2 local cones?

(charts to be glued)

Use matroid polytope version ($\widetilde{\mathcal{B}}(M)$ dual to loopless faces)



Subdivision of octahedron  $= P_M$ for $M = U(2,4)$ into 2 matroid polytopes.

- Each local cone is dual to the loopless part of a matroid polytope (by Key Prop)
- These polytopes glue together into a regular subdivision of the <u>d-hypersimplex</u> $\Delta(d,n)$ of $\mathbb{R}^n$ given by the matroid polytope pieces. = matroid subdiv of matroid polytope (valuated matroid)

$$\Delta(d,n) = \text{conv hull}\left(\sum_{i \in I} e_i \;:\; |I| = d\right) = P_{U(d,n)}$$

<u>Thm</u> [Speyer] A generic d-dim'l tropical linear space in $\mathbb{R}^n$ is dual to the loopless part of a matroid subdivision of $\Delta(d,n)$

- generic = matroid is $U(d,n)$.
- If it is non generic, we replace $\Delta(d,n)$ by P_M where M is a rank d matroid on [n]

<u>Note</u>: Constant coefficient case = subdivision is trivial.

<u>Prop</u>: If M = matroid of K_n, then $\widetilde{\mathcal{B}}(K_n)$ = space of phylogenetic trees on $n+1$ leaves
[Ardila-Klivans]

This justifies $\mathrm{Trop}\, Gr_\phi(2,4)/\!\!/\mathbb{L}$ =  = Tropical line in $\mathbb{R}^2$
(Lecture XIV) (tropical linear space $\widetilde{\mathcal{B}}(K_3)$)

Q: What about other matroid strata of $Gr(2,n)$?

<u>Q2</u>: Combinatorics of Tropical linear spaces?

<u>Speyer's f-vector conjecture</u> $\#$ i-dim'l bounded faces in $\mathrm{Trop}(M)/\mathbb{R}\cdot\mathbb{1}$ $\leq \dfrac{(n-i-1)!}{(d-i)!\,(n-d-i)!\,(i-1)!}$

- Proven for realizable matroids over $\mathbb{Q}\{\{t\}\}$ [Speyer: "A matroid invariant via the K-theory of the Grassmannian"]
- Open in general | Bound is sharp (= for "series-parallel matroids")
- From the conjecture: # i-dim'l faces in $\operatorname{Trop}(V)/\mathbb{R}\cdot 1 \leq \binom{n-i-1}{d-i}\binom{2n-d-1}{i-1}$.

§2 Moduli Space of Tropical linear spaces:

. Recall: d dim'l linear spaces $\rightsquigarrow$ realizable matroid of rank d

. points in $Gr_J(d,n)$ correspond to linear spaces with matroid J^c.

$$I(Gr_J(d,n)) = (I_{d,n} + \langle p_\sigma : \sigma \text{ not a bases of } J^c\rangle) \cap K[p_B^{\pm 1} : B \in J^c]$$

generated by $\left\{ \sum_{\substack{j \\ \sigma\cup j,\, \tau\smallsetminus j \in J^c}} \operatorname{sgn}(j,\sigma,\tau)\, p_{\sigma\cup j}\, p_{\tau\smallsetminus j} \;:\; \begin{array}{l} |\sigma| = d-1 \\ |\tau| = d+1 \\ |\tau\smallsetminus\sigma| \geq 3 \end{array} \right\}$

Tropicalize these generators to get:

$$\operatorname{trop} \mathcal{Pl}_{\sigma,\tau} = \max_j \{ w_{\sigma\cup j} + w_{\tau\smallsetminus j} \}$$

Def A point $\underline{w}$ in $\mathbb{R}^{|J^c|}/\mathbb{R}\cdot 1$ is a tropical Plücker vector for J^c if it satisfies all $\operatorname{trop} \mathcal{Pl}_{\sigma,\tau}$.

Def The Dressian of J^c is the set of all tropical Plücker vectors. Write it Dr_{J^c}

Notice: $\operatorname{Trop} Gr_J(d,n) \subseteq Dr_{J^c}$ & $=$ holds for $d=2$.

realizable trop. linear spaces w/ matroid J^c

(tropical) valuated matroids [Dress-Wenzel]

Prop: Points in Dr_{J^c} parametrize tropical linear spaces (= valuated matroids w/ matroid J^c)

How? Given w in Dr_{J^c}, we build L_w tropical linear space in $\mathbb{R}^{|J^c|}/\mathbb{R}\cdot 1$.

- $\tau \in \binom{[n]}{d+1}$, $rk(\tau) = d \rightsquigarrow L_\tau(w) =$ "$V(\operatorname{trop}(\sum_{j\in\tau} p_{\tau\smallsetminus j}\, x_j))$" $(*)$

$$= \{\underline{x} \mid \max_{j\in\tau} (w_{\tau\smallsetminus j} + X_j) \text{ attained twice}\}$$

[Think: $-\operatorname{val}(p_{\tau\smallsetminus j}) = w_{\tau\smallsetminus j} \rightsquigarrow$ comes from $Gr_J(d,n)$.]

- The tropical linear space $= L_w = \bigcap_\tau L_\tau(w)$ is dual to matroid subdivision of P_{J^c} defined by $-w_B$ [$ht(v_B) = -w_B$]

Prop: Any Trop Plücker vector gives a matroidal subdivision of P_{J^c}.

Structure Thm: L_w is a pure $(d-1)$-dimensional balanced (wt=1) contractible polyhedral complex in $\mathbb{R}^n/\mathbb{R}\cdot 1$, with recession fan $= \tilde{\mathcal{B}}(M)$ & degree $=1$ ($L_w \cap (p + \operatorname{trop} U_{(n-d+1,n)})$ is 1 pt. generic pt