Lecture XVIII: Structure Theorem for Tropical Varieties

So far: Given $I \subset K[x_1^{\pm}, \dots, x_n^{\pm}]$, we defined $\mathrm{Trop}(V(I)) = \bigcap_{f \in I} \mathcal{T}(V(f)) \subseteq \mathbb{R}^n$

[K valued field, $\Gamma_{val} = val(K^{\times}) \subset \mathbb{R}$]

↳ can use finitely many (tropical basis)

Saw (Kapranov's Thm): This set agrees with

(1) $\{ w \in \mathbb{R}^n : in_w I \neq \langle 1 \rangle \}$

(2) closure of $\{ (-val(y_1), \dots, -val(y_n)) : y \in V(I) \}$ in $\mathbb{R}^n$

when $\bar{K} = K$ & val on K is nontrivial

- Tropical Hypersurfaces: $\mathcal{T}(V(f))$ is the collection of codim-1 cells in the dual complex of the Newton subdivision of f.

GOAL: What is the structure of $\mathrm{Trop}(V(I))$? Characterizing properties?

Prop [Gröbner characterization] Let $I \subset K[x_1^{\pm}, \dots, x_n^{\pm}]$

Then, $\{ w \mid in_w I \neq \langle 1 \rangle \}$ is the support of a subcomplex of the Gröbner complex $\Sigma(I_{proj})$ and is thus the support of a Γ_{val}-rational polyhedral complex

Here: $I_{proj} := \langle f^h : f \in I \cap K[x_1, \dots, x_n] \rangle \subset K[x_0, \dots, x_n]$ (defining $\overline{V(I)} \subseteq \mathbb{P}^n$)

where $f^h(x_0, \dots, x_n) = x_0^N f\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right)$ for N minimal such that $x_0 \nmid$ (RHS).

Proof: • $\Sigma(I_{proj}) \subseteq \mathbb{R}^{n+1}/\mathbb{R}\cdot\mathbb{1}$ was constructed in Lecture X & was shown to be Γ_{val}-rat'l polyhedral complex.

- We identify $\mathbb{R}^{n+1}/\mathbb{R}\cdot\mathbb{1} \simeq \mathbb{R}^n$ via $w \longmapsto (w_i - w_0)_i$ & $(0, w) \longleftarrow w$.

- Prop from Lecture XI: $\underset{\underset{K[x_1^{\pm}, \dots, x_n^{\pm}]}{\cap\!\!\!|}}{in_w I} = \langle \underset{\underset{K[x_0, \dots, x_n]}{\cap\!\!\!|}}{in_{(0,w)} I_{proj}} \big|_{x_0 = 1} \rangle$ Furthermore if

$f \in in_w I$, then $f = x^u g$ for $g = h(1, x_1, \dots, x_n)$ & $h \in in_{(0,w)} I_{proj}$.

So $\{ w : in_w I \neq \langle 1 \rangle \} = \{ w \mid in_{(0,w)}(I_{proj})$ contains no monomials $\}$

$\Rightarrow$ (RHS) is a union of closures of cells of $\Sigma(I_{proj})$ because having a monomial is an open condition in $0 \times \mathbb{R}^n$ since $in_{(0,v)}\, in_{(0,w)} I_{proj} = in_{(0, w+\varepsilon v)} I_{proj}$ for $\varepsilon \ll 1$. $\square$

NOTE: • This is not the only structure we can put on $\mathrm{Trop}(V(I))$ when $K = \bar{K}$, val nontrivial

- Construction is sensitive to coordinate changes in $(K^{\times})^n$ (see HW 2)
- $\Phi : (K^{\times})^n \longrightarrow (K^{\times})^m$ monomial map assoc to $A \in \mathbb{Z}^{m \times n}$ $\mathrm{Trop}(\Phi(X)) = A\, \mathrm{Trop}(X)$

Obs: If $A\cdot : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is not injective, then (RHS) does not inherit a polyhedral structure from Trop(X).

Furthermore, cells in the image need not intersect in faces, but we can always refine to get a polyhedral structure (because the (LHS) has one)

Push forward formula of [Sturmfels-Tevelev] moves weight/multiplicities from Trop(X) to $\text{Trop}(\overline{\Phi(X)})$.

(*) see page 4.

Q: What else can we say about Trop(V(I))? Since $\text{Trop}(X \cup X') = \text{Trop}(X) \cup \text{Trop}(X')$ we restrict to irreducible varieties

Structure Thm: Let $X \subset (K^\times)^n$ be irreducible of dimension $= d$. Then:

(1) Trop(X) is the support of a pure d-dim'l Γ_{val}-rat'l polyhedral complex in $\mathbb{R}^n$ [Bieri-Groves]

(2) Trop(X) is connected through codimension 1

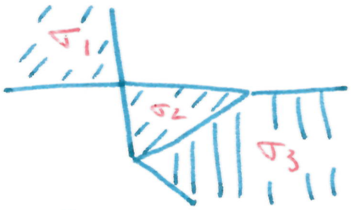
(3) Trop(X) can be equipped with weights/multiplicities on its maximal cells & with them it becomes balanced at all its codimension 1 cells.

Note: • (2) Build a graph G with vertices $\longleftrightarrow$ mxl cells $[i \longleftrightarrow \sigma_i]$

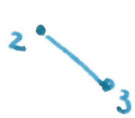
edges (i,j) if $\sigma_i \cap \sigma_j$ along a codim-1 face.

Condition (2) means G is a connected graph

Non-example:



G :

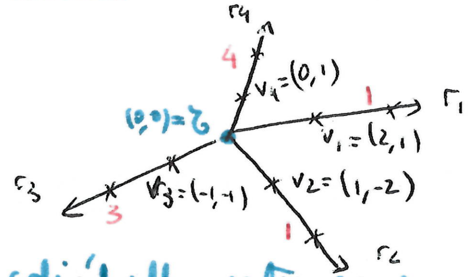


$\rightsquigarrow$ Algorithm in GFan.

• For (3), start with balancing:

Given $\Sigma \subset \mathbb{R}^n$ a pure d dim'l rational polyhedral complex, want to define balancing at codim-1 cells. We divide the definition in 3 cases:

CASE 1 : $d=1$ Σ fan.



1 codim'l cell = vertex (0,0)

• For each ray $r_i \in \Sigma^{(1)}$ (→ 1-skeleton), let v_i be the first lattice pt in r_i (we have one because r_i has rat'l slope)

• Weight / Multiplicity on ray r_i $= m_i \in \mathbb{Z}_{>0}$

(Eg: $m_1 = m_2 = 1$, $m_3 = 3$, $m_4 = 4$

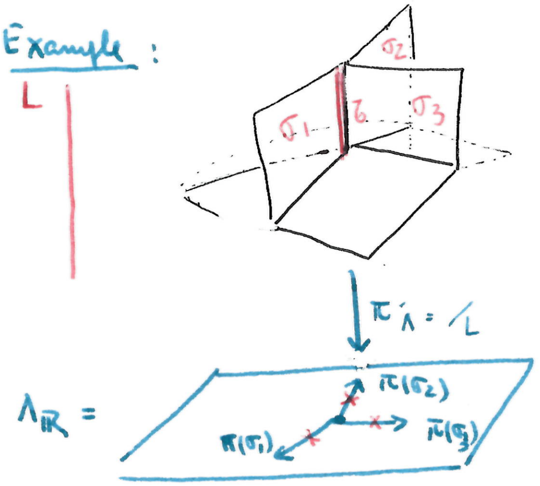
• Zero Tension / Balancing at (0,0) : $\sum_i m_i v_i = 0$.

(Ex is balanced)

CASE 2 : Any $d \geq 1$ Σ fan.

- $\Sigma^{(d)} = \{\sigma_1, \ldots, \sigma_r\}$ mxl cones. Assing weight $m_\sigma \in \mathbb{Z}_{>0}$ to each $\sigma \in \Sigma^{(d)}$
- We define balancing at a codim-1 cell of Σ by reducing to CASE 1.

Example :



given $\tau \in \Sigma^{(d-1)}$, let L = linear space of $(d-1)$ spanned by τ in $\mathbb{R}^n$

τ is a rat'l cone, so $L_{\mathbb{Z}} = L \cap \mathbb{Z}^n$ is free of rk $= d-1$.

It is a saturated sublattice of $\mathbb{Z}^n$ & has a complementary lattice $\Lambda \simeq \mathbb{Z}^{n-d+1}$ with $L_{\mathbb{Z}} \oplus \Lambda = \mathbb{Z}^n$.

Write $N(\tau) = \mathbb{Z}^n / L_{\mathbb{Z}} \simeq \Lambda$.

given $\sigma \gneq \tau$ mxl cone, we get

$\frac{\sigma + L}{L}$ is a rat'l 1-dim'l cone in $N(\tau) \otimes_{\mathbb{Z}} \mathbb{R} \cong \Lambda_{\mathbb{R}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}^{n-d+1}$,

so we can find $v_\sigma :=$ first lattice point in $\frac{\sigma+L}{L}$.

Define : The fan Σ is <u>balanced</u> at τ if $\sum_{\sigma \gneq \tau} m_\sigma v_\sigma = 0$.

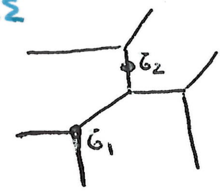
Equivalently, a lift of v_σ to σ (call it u_σ) satisfies $\sum_{\sigma \gneq \tau} m_\sigma u_\sigma \in L$.

[From CASE 1 : $\{\frac{\sigma+L}{L} : \sigma \gneq \tau\}$ form a fan of dim 1 in $\Lambda_{\mathbb{R}}$ & balanced at $\underline{0}$ at $\underline{0} = \frac{\tau + L}{L}$]

CASE 3 : Any d, Σ polyhedral complex. Work with local cones (aka stars!)

given τ cell in Σ, consider $\text{Star}_\Sigma(\tau) = \bigcup_{\sigma \geq \tau} \overline{\sigma}$ where $\overline{\sigma} := \{\lambda(x-y) : \lambda \geq 0, x \in \sigma, y \in \tau\}$

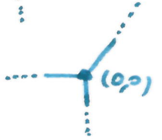
Example :



$\text{Star}_\Sigma(\tau_2) =$

= parallel v. sp to τ_2.

$\text{Star}_\Sigma(\tau_1) =$



Prop : $\text{Star}_\Sigma(\tau)$ is a polyhedral fan.

Def : The complex Σ is balanced at τ if $\text{Star}_\Sigma(\tau)$ is balanced at $\overline{\tau}$.

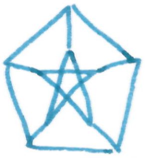
NEXT TIME :
- Construct multiplicities from Geometry of X (and $w_\omega t$)
- Discuss [Bieri-Groves] result

(*) What about other structures for Trop(X), i.e not coming from Gröbner structure?

Example 1 in HW2 (under change of coordinates)

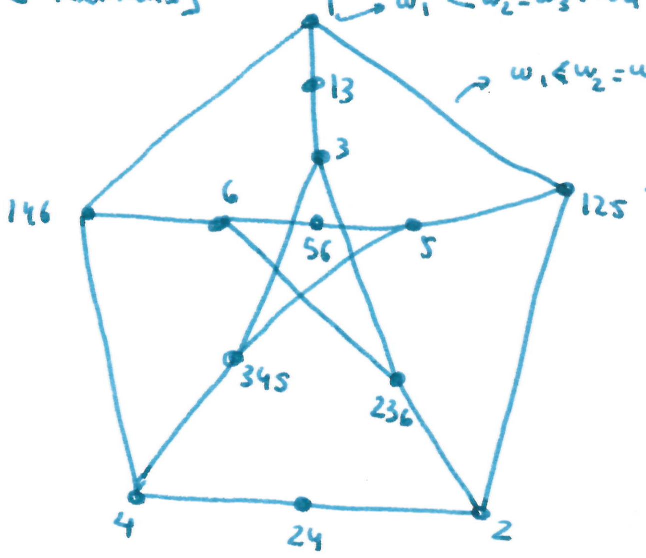
Example 2 Trop Linear spaces admit various structures:

- $\mathrm{Trop}\, Gr_0(2,5)$ = cone over the Petersen graph



- Lecture XVII: $\mathrm{Trop}\, Gr_0(2,5) \underset{\hookrightarrow \text{ same support}}{=} \tilde{\mathcal{B}}(K_4)$ Bergman fan of K_4

Concrete computations give: $\tilde{\mathcal{B}}(K_4)$ = cone over the subdivided Petersen graph: [Ardila-Klivans]

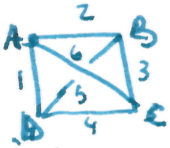


1 ↦ $w_1 < w_2 = w_3 = w_4 = w_5$

↗ $w_1 < w_2 = w_5 < w_3 = w_4 = w_6$

125 → $w_1 = w_2 = w_5 < w_3 = w_4 = w_6$

K_4 :



$\tilde{\mathcal{B}}(K_4) \subseteq \mathbb{R}^{\binom{4}{2}=6}$

Ground set = $\{\underset{1}{AD}, \underset{2}{AB}, \underset{3}{BC}, \underset{4}{CD}, \underset{5}{BD}, \underset{6}{AC}\}$

Basis = Spanning trees : {162, 263, 354, 154, 123, 234, 341, 412}