Lecture XIX: Tropical multiplicities & Bieri-Groves Thm

GOAL: Define tropical multiplicities

- Take $X \subset (K^*)^n$ defined by $I \subset K[x_1^\pm, \ldots, x_n^\pm]$ & Take Gröbner polyhedral structure on $\mathrm{Trop}(X)$ defined by $\Sigma(I_{proj}) \cap \{x_0 = 0\}$. In particular:

  $\forall\, w \in \mathrm{relint}(\sigma)$: $\mathrm{in}_w(I)$ is constant for any cell σ in $\mathrm{Trop}(X)$.

  Define $m_w :=$ # irreducible components of $\mathrm{in}_w I \subseteq \mathbb{k}[x_1^\pm, \ldots, x_n^\pm]$ counted with mult

For this, we need a primary decomposition. Set $S = \mathbb{k}[x_1^\pm, \ldots, x_n^\pm]$

Def: An ideal $Q \subset S$ is *primary* if $fg \in Q \Rightarrow f \in Q$ or $g^m \in Q$ for some $m > 0$.

FACT: $P = \mathrm{Rad}(Q)$ is a prime ideal if Q is primary: call it P-primary.

Primary Decomp: given $J \subset S$ ideal $\quad J = \bigcap_{i=1}^{s} Q_i \quad$ for $Q_i = P_i$-primary & $P_i \neq P_j$ & no term can be removed from the intersection

- $\mathrm{Ass}(J) = \{P_i : 1 \leq i \leq s\}$ associated primes
- Decomp is not unique except for Q_i's associated to minimal primes $P_i \supset J$ over J.

Geometrically: non-embedded components correspond to minimal primary components.

Write $\mathrm{Ass}(J)_{min} = \{P_i \in \mathrm{Ass}(J)$ minimal over $J\}$

Def: The *multiplicity* of $P_i \in \mathrm{Ass}(J)_{min}$ is $\mathrm{mult}(P_i, J) = \mathrm{length}\left((S/Q_i)_{P_i}\right) \in \mathbb{Z}_{>0}$

where $\ell(M)$ is the length of the S_{P_i}-module M, i.e. the length of the longest chain of submodules of $M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_s$

Example 1 $f = \alpha \prod_{i=1}^{r} (x - \lambda_i)^{m_i}$ & $J = \langle f \rangle$ with $\alpha, \lambda_i \in \mathbb{k}$.

$\Rightarrow \mathrm{Ass}(J) = \{\langle x - \lambda_i \rangle : i = 1, \ldots, r\}$ & all are minimal | $Q_i := \langle (x-\lambda_i)^{m_i} \rangle$

$\mathrm{mult}(\langle x - \lambda_i \rangle, J) = m_i$. (chain $\langle x-\lambda_i \rangle \supset \langle x - \lambda_i \rangle^2 \supset \cdots \supset \langle x-\lambda_i \rangle^{m_i} = \{0\}$ in $(S/Q_i)_{P_i}$)

& $\sum_{i=1}^{r} \mathrm{mult}(\langle x - \lambda_i \rangle, J) = \sum_{i=1}^{r} m_i = \deg(f)$

Def: Given $\mathrm{Trop}(I)$ with its Gröbner structure, & $w \in \mathrm{relint}(\sigma)$ for a given cell σ of $\mathrm{Trop}(I)$, we define:

$$\mathrm{mult}(\sigma) = \sum_{P \in \mathrm{Ass}(\mathrm{in}_w I)_{min}} \mathrm{mult}(P, \mathrm{in}_w I) \in \mathbb{Z}_{>0}$$

Note: Definition works for any cell in $\mathrm{Trop}(I)$, not only maximal.

Example 2 I is principal $= \langle f \rangle$: σ max'l cell is dual to an edge of the Newton subdiv. of f.

Say the edge connects x^α & x^β so $in_\omega(f) = x^\alpha \sum_{i=0}^m a_i \, x^{v \cdot i} = x^\alpha \sum_{i=0}^m a_i (x^v)^i$ deg m poly. in (x^v).

write v = primitive vector in direction $\beta - \alpha$, with $\beta - \alpha = m \cdot v$

So $\mathrm{mult}(\sigma) = m$ = lattice length of dual edge.

§1 Alternative definition (better for computations)

Recall given $\tau \in \Sigma$: $\mathrm{Star}_\Sigma \tau = \bigcup_{\sigma \geq \tau} \overline{\sigma}$ $\quad \overline{\sigma} = \{\lambda(x-y) : \lambda > 0, x \in \sigma, y \in \tau\}$

Rmk: If σ is a max'l cell of dim $= d$, then $\mathrm{Star}_\Sigma \sigma = \mathbb{R}\langle \sigma \rangle$ is a d-dim'l linear space L
$L \cap \mathbb{Z}^n = L_\mathbb{Z}$ is $\mathbb{Z}$-spanned by $\{v_1, \ldots v_d\}$

- given $w \in \mathrm{relint}(\sigma)$, $in_\omega I$ is homogeneous wrt $A = \begin{pmatrix} v_1 \\ \vdots \\ v_d \end{pmatrix} \in \mathbb{Z}^{d \times n}$.

In particular, we can pick a complementary rank $(n-d)$ lattice $\Lambda = L^\perp \cap \mathbb{Z}^n$ so $L_\mathbb{Z} \oplus \Lambda = \mathbb{Z}^n$

- $\Lambda = \mathbb{Z}\langle w_1, \ldots, w_{n-d} \rangle$ so $in_\omega I \subseteq \mathbb{K}[x_1^\pm, \ldots, x_n^\pm]$ is generated by polynomials in $\mathbb{K}[(x^{w_1})^\pm, \ldots, (x^{w_{n-d}})^\pm] = S'$ (Exercise)

Eg $\sigma = \mathbb{R}\langle (1,1) \rangle$ $\quad v_1 = (1,1)$, $w_1 = (1,-1)$
$x + y = \left(\frac{x}{y} + 1\right) y \Rightarrow \frac{x}{y} + 1 \in \mathbb{K}\left[\frac{x}{y}\right]$

Lemma: Assume I is prime, dimension d. Then $\mathrm{mult}(\sigma) = \dim_\mathbb{K} \left(S' / in_\omega I \cap S' \right)$

Proof: After an automorphism of tori, we may assume $v_1 = e_1, \ldots, v_d = e_d$
$w_1 = e_{d+1}, \ldots, w_{n-d} = e_{n-d}$
(call $y_i = x^{v_i}$, $z_j = x^{w_j}$)

By the Remark above: $in_\omega I$ has a generating set $\{f_1, \ldots, f_s\}$ non containing the variables $x_{d+1}, \ldots, x_n$. Write $in_\omega I = \bigcap_{i=1}^r Q_i$ primary decomp.

By construction Q_i is also generated by polynomials in S' for all $i = 1, \ldots, r$

$\Rightarrow in_\omega(I) \cap S' = \bigcap_{i=1}^r (Q_i \cap S')$ is a primary decomposition of $J = in_\omega I \cap S'$

- $\dim I = d$, so by flatness $\dim in_\omega I = d$ $\Rightarrow \dim (in_\omega I) \cap S' = 0$ (*)
gens in S'
$\Downarrow$
all $P_i \cap S'$ are minimal (max'l ideals in S')

- $\mathrm{mult}(P_i, in_\omega I) = \mathrm{mult}(P_i \cap S', in_\omega I \cap S')$
$= \underset{(Q_i \cap S')}{\text{colength}} = \dim_\mathbb{K} \left(S' / Q_i \cap S' \right)_{P_i} = \dim_\mathbb{K} \left(S' / Q_i \cap S' \right)$

Claim: (dim $= 0$) $\sum_i \dim_\mathbb{K} \left(S' / Q_i \cap S' \right) = \dim_\mathbb{K} S' / in_\omega \cap S'$ (by "Chinese Remainder Thm") □

Corollary (*): $in_\omega I$ is a union of f. many d-dim'l tori if $w \in \mathrm{relint}(\sigma)$ & σ max'l cell in $\mathrm{Trop}(I)$

§2 Bieri-Groves Thm:

Thm [BG] $I \subset K[x_1^{\pm}, \ldots, x_n^{\pm}]$ irreducible, $\dim = d$, then $\mathrm{Trop}(X)$ is pure of $\dim = d$.

Lemma 1. Given $w \in \mathrm{relint}(\sigma)$, $\sigma \in \mathrm{Trop}(I)$, then $\mathrm{Trop}(V(\mathrm{in}_w I)) = \mathrm{Star}_{\mathrm{Trop}\, I}(\sigma)$.

Pf/ $\mathrm{Star}_{\mathrm{Trop}\, I}(\sigma) = \{ v \in \mathbb{R}^n : \underbrace{\mathrm{in}_v(\mathrm{in}_w I)}_{\mathrm{in}_{w+\varepsilon v}(I)} \neq \langle 1 \rangle \}$. $\varepsilon \ll 1$ □

Lemma 2 $\dim \mathrm{Trop}\, I \leq d$

Pf/ After working with $K = \overline{K}$ & nontrivial valuation, we know Γ_{val} is dense in $\mathbb{R}$

Take $w \in \mathrm{relint}(\sigma)$ & σ a maxl cell of $\dim = k$, $\mathrm{Star}_{\mathrm{Trop}\, I}\sigma = \mathbb{R}\langle \sigma \rangle = L$

Remark from page 2 ensures that $\mathrm{in}_w I$ is generated by polynomials in $\mathbb{K}[w_1^{\pm}, \ldots, w_{n-k}^{\pm}]$ where $\langle w_1, \ldots, w_{n-k} \rangle = L^{\perp}_{\mathbb{Z}}$. $L_{\mathbb{Z}} = \langle v_1, \ldots, v_k \rangle$

$\Rightarrow k \leq \dim(\mathrm{in}_w I) = \dim I = d$

$\mathfrak{P}' \subset \mathfrak{P}' + \langle v_1 - 1 \rangle \subsetneq \cdots \subsetneq \mathfrak{P}' + \langle v_1 - 1, \ldots, v_k - 1 \rangle$ chain of prime ideals.

$\mathfrak{P}'$: prime in $\mathbb{K}[w_1^{\pm}, \ldots, w_{n-k}^{\pm}]$ containing the gens of $\mathrm{in}_w I$.

Proof Thm: Need to show all maxl cells in $\mathrm{Trop}\, I$ have $\dim \geq d$.

As in Proof of Lemma 2. $\mathrm{in}_w I$ generated by polynomials in $\mathbb{K}[w_1^{\pm}, \ldots, w_{n-k}^{\pm}] = S'$

Write $J = \mathrm{in}_w I \cap S'$

Claim: $\mathrm{Trop}(V(J)) = \{0\}$

After changing coordinates on the ambient torus, we may assume $v_1 = e_1, \ldots, v_k = e_k$, $w_1 = e_{k+1}, \ldots, w_{n-k} = e_n$

By double inclusion:

($\subseteq$) Pick $v' \in \mathrm{Trop}\, V(J) \cap \Gamma_{val}^{n-k}$ & $v = (0, v')$ (Recall Γ_{val} dense in $\mathbb{R}$)

If $v' \neq 0$, then $v \notin L = \mathbb{R}\langle\sigma\rangle$ & so $\mathrm{in}_v(\mathrm{in}_w I) = \mathrm{in}_{w+\varepsilon v} I = \langle 1 \rangle$

because $w + \varepsilon v \notin$ & σ is maxl. Since $\mathrm{in}_w I$ is homog & $v \in \Gamma_{val}^n$

by Key Prop from Lecture 8, $1 = \mathrm{in}_v(f)$ for some f in $\mathrm{in}_w I$.

Furthermore, we can take f in J. We conclude $1 = \mathrm{in}_{v'} f$ Contradiction! ($\Rightarrow v' = 0$)

($\supseteq$) $1 \notin \mathrm{in}_w I$ so $\mathrm{in}_0 J = J \neq \langle 1 \rangle$ so $\{0\} \subseteq \mathrm{Trop}(V(J))$.

By Lemma 3 (below), $V(J)$ is finite so $\underbrace{\dim(\mathrm{in}_w I)}_{= d} \leq k$. so $k \geq d$ □

Lemma 3: If $\mathrm{Trop}(I)$ is finite, then I is ~~also~~ finite

Proof: By induction on n

- Base case: $n=1$ $\quad \mathrm{Trop}(K^*) = \mathbb{R}$ & all proper subvarieties of K^* are finite
- Inductive Step: If I is principal, we know $\mathrm{Trop}(I)$ is not finite.

So we can assume $\dim I < n-1$ $\quad$ Write $X = V(I)$

(*) Take a projection $\pi : (K^*)^n \longrightarrow (K^*)^{n-1}$ with $Y = \overline{\pi(X)}$ & $\dim Y = \dim X$

Up to changing coordinates, we may assume $\pi = \pi_n$ $\quad \underline{x} \longmapsto (x_1 \cdots x_{n-1})$.

But $\mathrm{Trop}(Y) \underset{\text{Lecture XVIII}}{=} \pi_n(\mathrm{Trop}\, X)$ so it's a finite set of pts in $\mathbb{R}^{n-1}$

By the inductive hypothesis, Y is then finite.

Claim: $\lambda e_n \notin \mathrm{Trop}(I)$, so we must have some $f = 1 + \sum_{i=1}^{s} f_i x_n^i \in I$

with $f_i \in K[x_1^{\pm}, \ldots, x_{n-1}^{\pm}]$ on $f_s \neq 0$.

So each element $y \in Y$ has at most s preimages in X under π, so X is finite $\square$

To find (*) we use the following proposition (essentially Noether Normalization in the Laurent setting).

Prop: Let $X \subset (K^*)^n$ & $m \geq \dim(X)$. There exists a morphism $\varphi : (K^*)^n \longrightarrow (K^*)^m$ where $Y = \overline{\varphi(X)} = \varphi_{(X)}$ & has $\dim Y = \dim X$.

Furthermore, when $n > m$, we can change coordinates so that φ is the projection onto the first m coordinates & the ideal I defining X is generated by polynomials in $x_{m+1}, \ldots, x_n$ whose coefficients are monomials in $x_1, \ldots, x_m$.

Proof: We give Noether Normalization on $K[x_1^{\pm}, \ldots, x_n^{\pm}]$

By induction on $n-m$:

- Base case $n-m=0$ take $\pi = id$.
- Inductive step $n-m>0$ so $I \neq 0$.

Given $l \in \mathbb{N}$ we define a invertible monomial map $\phi_l^* : (K^*)^n \longrightarrow (K^*)^n$

$\phi_l^*(x_1) = x_1 x_n^{l}$, $\phi_l^*(x_2) = x_2 x_n^{l^2}$, $\ldots$, $\phi_l^*(x_i) = x_i x_n^{l^i}$, $\ldots$, $\phi_l^*(x_n) = x_n$ $\quad 1 \leq i \leq n-1$

Aim to write exponents of generators of I in base l where all monomials have distinct degrees in x_n. We can do this for $l \gg 0$.

Pick a set generating I : $\{f_1, \dots, f_s\}$. Pick $\ell \gg 0$ so that

$$g_i = \phi_\ell^*(f_i) = f(x_1 x_n^{\ell}, \dots, x_i x_n^{\ell^i}, \dots, x_{n-1} x_n^{\ell^{n-1}}, x_n)$$

has monomials with distinct degrees in x_n. Write $g_i = \sum_{j=0}^{S_i} a_{ij}(x_1, \dots x_{n-1})\, x_i^j$ $(a_{i S_i} \neq 0)$ monomial! $(**)$

Since ϕ^* is invertible, we replace I with $J = \phi^*(I)$ & prove the Prop. for J.

<u>Claim</u>: $\pi : (K^\times)^n \longrightarrow (K^\times)^{n-1}$ $\quad (x_1, \dots, x_n) \longmapsto (x_1, \dots, x_{n-1})$ [works for $m = n-1$]

satisfies $\dim(X) = \dim(\overline{\pi(X)}) = m$. & furthermore $\overline{\pi(X)} = \pi(X)$.

Pf/. • $\overline{\pi(X)}$ is defined by $I \cap K[x_1^{\pm}, \dots, x_{n-1}^{\pm}]$

• $\overline{\pi(X)} \smallsetminus \pi(X) \subseteq V(\langle \{ a_{iS_i}(x_1, \dots, x_{n-1}) : i = 1, \dots, s \} \rangle) = \emptyset$.

$\hookrightarrow$ monomial in $x_1, \dots, x_{n-1}$ by $(**)$
so empty subvariety of $(K^\times)^{n-1}$!

• To prove the dimension statement, note that having a monomial leading term for the generators means we can get a set of generators that are monic in x_n.

so $K(X)$ / $K(\pi(X))$ is a finite extension [Why?: $K[X] = K[x_1^{\pm}, \dots, x_n^{\pm}]/I$ is generated by x_n as a $K[\pi(X)]$-algebra.]

we conclude: $\operatorname{tr\,deg}(K(X), K) = \operatorname{tr\,deg}(K(\pi(X)), K)$ □.

$\qquad\qquad \| \qquad\qquad\qquad\qquad \|$
$\qquad\quad \dim X \qquad\qquad\quad \dim(\pi(X))$

By induction on $(n-1) - m$ we find $\varphi : (K^\times)^{n-1} \longrightarrow (K^\times)^m$ with the desired properties for $\pi(X)$.

The composition $\Psi = \varphi \circ \pi : (K^\times)^n \longrightarrow (K^\times)^m$ gives the desired map. □

(precomposed with $(\phi_\ell^*)^{-1}$).