Lecture XX: Abstract Tropical Varieties & their structure

Last Time: Tropicalizations of d-dim'l irreducible subvarieties X of $(K^*)^n$ are pure Γ_{val}-rational polyhedral complexes of dim d in $\mathbb{R}^n$, balanced on codim 1 cells & connected in codim 1.

Q1: What happens when we change the valuation on K? **Eg:** Take trivial valuation

Q2: Can we use this to define abstract tropical varieties? • [Mikhalkin] • Eg ~~trop~~ linear spaces from non-realizable valuated matroids

§1: Constant coefficients & Recession Fans:

• By def, $\mathrm{Trop}(V(f))$ is a fan when K has trivial valuation.

$\leadsto$ If X is defined by I $\quad \mathrm{Trop}(X) = \bigcap_{f\in I} \mathrm{Trop}(V(f))$ is again a fan.

• If valn is non-trivial, $\mathrm{Trop}(X)$ is a complex. The behavior "at ∞" is determined by its recession fan

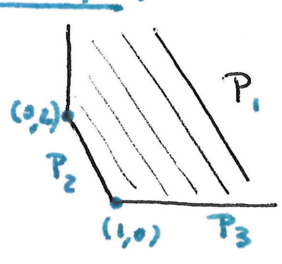
Def: Given a polyhedron $P = \{x : Ax \leq b\}$, the recession cone of P is

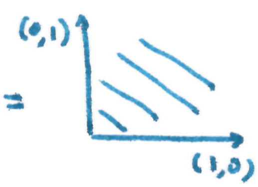
$$\mathrm{rec}(P) = \{x : Ax \leq 0\} = \bigcap_{p\in P} \mathrm{rec}_p P \quad \& \quad \mathrm{rec}_p P = \{u\in\mathbb{R}^n \mid p+\lambda u \in P \ \forall \lambda\geq 0\}$$

Note: • It is the unique cone satisfying $P = \mathrm{rec}(P) + Q$ for some polytope Q. Here, Q need not be unique.

• $\mathrm{rec}(P)$ is the cone dual to the support of the normal fan of P.

Example:

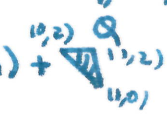


• $\mathrm{rec}(P_1) =$ 

$\quad (P_1 = \mathrm{rec}(P_1) +$ 

$)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \overset{\shortparallel}{Q}$

$= \mathrm{rec}(P_1) +$ 

• $\mathrm{rec}(P_2) = \{(0,0)\}$ $\quad$ & $P_2 = Q_2$)

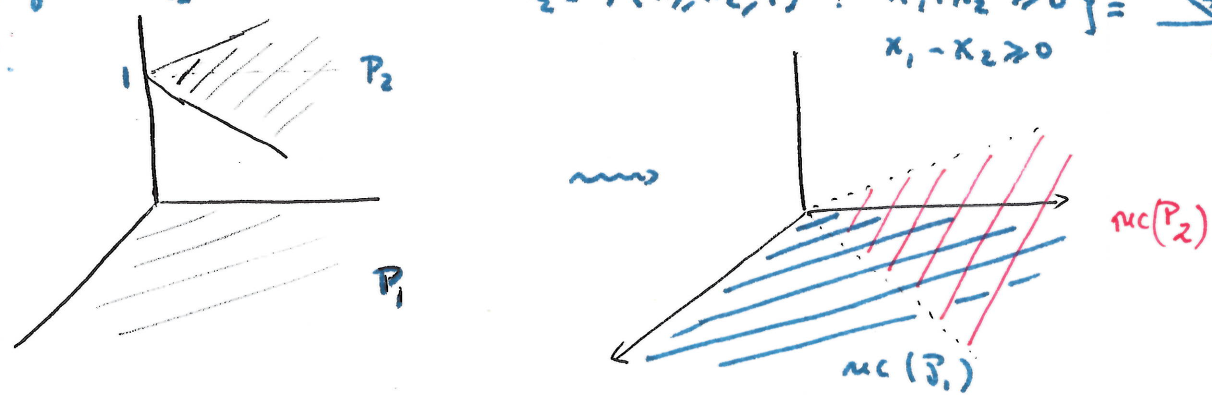
• $\mathrm{rec}(P_3) = \xrightarrow[(1,0)]{}$ $\quad$ & $Q_3 = \{(1,0)\}$)

Def 2: If $\Sigma \subset \mathbb{R}^n$ is a polyhedral complex, the recession fan $\mathrm{rec}(\Sigma)$ of Σ is the union $\bigcup_{P\in\Sigma} \mathrm{rec}(P)$ (as a set).

Note • $\mathrm{rec}(\Sigma)$ is the support of a polyhedral fan, but the structure is NOT canonical in particular, we need not have a coarsest fan structure [Burgos Gil-Sombra]

• If Σ satisfies • Σ connected
• $\forall P\in\Sigma : \mathrm{rec}_p P = \mathrm{rec}\, P$
, then $\{\mathrm{rec}(P) : P\in\Sigma\}$ gives a fan structure on $\mathrm{rec}(\Sigma)$

Example $\Sigma = P_1 \sqcup P_2$ $\quad P_1 = \{(x_1, x_2, 0) : x_1, x_2 \geq 0\} = \mathbb{R}^2_{\geq 0} \times \{0\}$
[Bungmfil-Sandra]
$P_2 = \{(x_1, x_2, 1) : x_1 + x_2 \geq 0, \; x_1 - x_2 \geq 0\} =$  $\times \{1\}$



Lemma: If $P \cap Q \neq \emptyset$ & both are polyhedra in $\mathbb{R}^n$, then $\operatorname{rec}(P \cap Q) = \operatorname{rec}(P) \cap \operatorname{rec}(Q)$.

Proof: Write $P = \{\underline{x} : A\underline{x} \leq \underline{b}\}$
$Q = \{\underline{x} : C\underline{x} \leq \underline{d}\}$ $\rightsquigarrow P \cap Q = \left\{ \underline{x} \; \begin{pmatrix} A \\ C \end{pmatrix} \underline{x} \leq \begin{bmatrix} b \\ d \end{bmatrix} \right\} \neq \emptyset$ by assumption

By definition $\operatorname{rec}(P) = \{\underline{x} : A\underline{x} \leq 0\}$, $\operatorname{rec}(Q) = \{\underline{x} : C\underline{x} \leq 0\}$.
& $\operatorname{rec}(P \cap Q) = \{\underline{x} : \begin{pmatrix} A \\ C \end{pmatrix} \underline{x} \leq 0\}$. The result follows! □

Thm: $X \subset (K^*)^n$, then $\operatorname{rec}(\operatorname{Trop} X) = \operatorname{Trop}_{\text{trivial}}(X)$ as sets!

Tropicalization w.r.t the given valuation on K. $\rightsquigarrow$ Tropicalization w.r.t trivial valuation on K

Proof: (1) Assume X is a hypersurface, say $X = V(f)$. Take $P \leq \operatorname{Trop}(f)$ cell.
- If P is bounded, then $\operatorname{rec}(P) = \{0\}$
- If P is unbounded, then P is dual to an exterior subdivided face of $NP(f)$, say $\underset{\substack{\text{face in} \\ \text{subdiv}}}{F} \leq \underset{\substack{\text{nonface in} \\ NP(f)}}{\mathcal{F}}$

$\Rightarrow \operatorname{rec}(P)$ = dual vectors to the face $\mathcal{F} \leq NP(f)$.

We conclude $\operatorname{rec}(\operatorname{Trop}(x))$ = codim-1 faces of $\mathcal{N}(\mathcal{F}) = \operatorname{Trop}_{\text{trivial}}(V(f))$.

(2) For the general case, use the Lemma & pick a tropical basis for the ideal I defining X wrt both valuations: the given valn on K & the trivial one. Call it B (finite!)

By definition: $\operatorname{Trop}_{\text{Trivial}}(X) = \bigcap_{f \in B} \operatorname{Trop}_{\text{Triv}}(V(f)) \underset{\text{Case (1)}}{=} \bigcap_{f \in B} \operatorname{rec} \operatorname{Trop}(V(f)) =$

$\underset{(*)}{=} \operatorname{rec}\left(\bigcap_{f \in B} \operatorname{Trop} V(f)\right) = \operatorname{rec}(\operatorname{Trop} X)$

For $(*)$: use Lemma & decompose each $\operatorname{Trop} V(f)$ into Gröbner cells, & $|\operatorname{rec}(P \cup Q)| = |\operatorname{rec} P| \cup |\operatorname{rec} Q|$. □

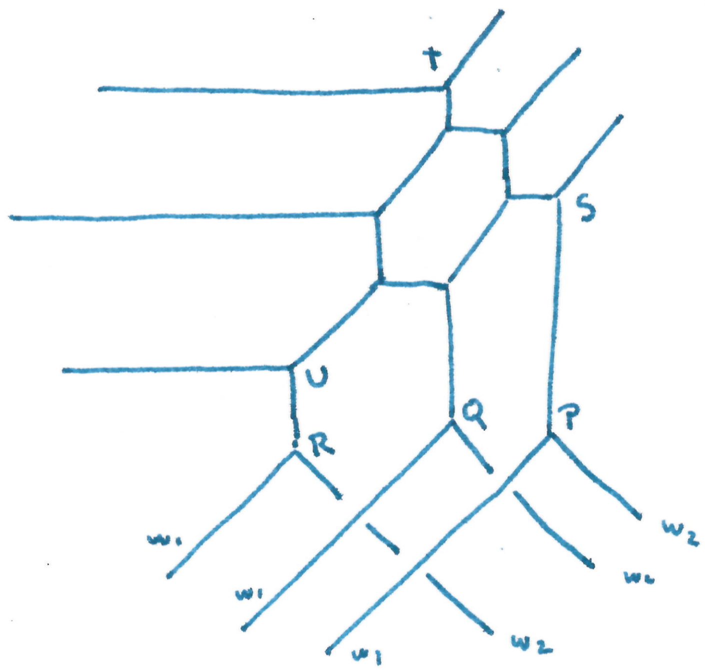
§2 Abstract tropical varieties

Def: An abstract tropical variety in $\mathbb{R}^n$ over Γ_{val} is a pure Γ_{val}-rat'l connected polyhedral complex, with $\mathbb{Z}_{>0}$-weights on max'l cells & balanced in codim-1 cells

WARNING: Not every abstract Trop variety is realizable (ie $= \operatorname{Trop}(X)$ for $X \subset (K^*)^n$)

Example (Mikhalkin) Tropical curves of genus 1 & degree 3 in $\mathbb{R}^3$

↳ 3 unbounded edges on 4 different directions: $-e_1$, e_1+e_2, w_1, w_2 with $w_1 + w_2 = -e_2$

$C =$ 

- Points above P, Q, R in a 2-dim'l plane
- Degrees of freedom for C

(1) Translation in $\mathbb{R}^3$ (3 dim)

(2) vary side lengths of hexagon (4 dim's, hex is closed!)

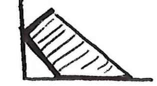
(3) Vary positions of P, Q, R, S, T, U along trop lines in $\mathbb{R}^2$ (6 dims)

$\Rightarrow$ C varies in a 13-dim'l family

BUT: Space of genus 1 curves of degree 3 in $\mathbb{R}^3$ is 12-dim'l $\Rightarrow$ most curves are non realizable!
In particular, if P, Q, R are not in a tropical line in $\mathbb{R}^3$, then C can't be realizable
(Because we would have $C = \mathrm{trop}(X)$ & X ~~must~~ lie in a hyperplane $\overline{X}$: degree 3 curve in $\mathbb{P}^3$)

Prop: Tropical hypersurfaces in $\mathbb{R}^n$ are realizable.

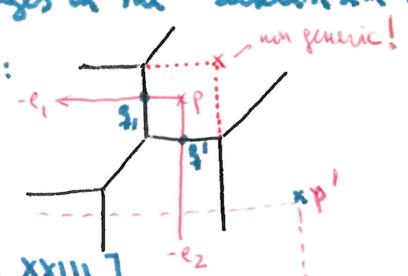
Proof: Ray-shooting algorithm to build a polytope P with a subdivision dual to the tropical hypersurface Σ. Here, P will be unique up to translation, we pick the one in the $\mathbb{R}^n_{\geq 0}$, meeting all coord. hyperplanes



- Vertices of P = connected components of $\mathbb{R}^n \setminus \Sigma$
  $v \longleftrightarrow C_v$
- Vertex $v = (v_1, \dots, v_n)$ from a generic pt p in $\{C_v\}$
  - $p + \mathbb{R}_{\geq 0}(-e_j)$ meets Σ at finitely many pts $\{q_1, \dots, q_s\}$ in mxl cells of Σ
  - each pt q_i comes from a cell σ_i & is counted with mult:
    $\mathrm{wt}(q_i) = m(\sigma_i) \cdot |\det(w_1 \cdots w_{n-1}\ {-e_j})|$ where $\mathrm{star}_\Sigma \sigma_i \cap \mathbb{Z}^n = \mathbb{Z}\langle w_1, \dots, w_{n-1}\rangle$

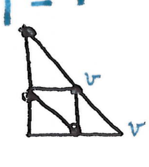
  $\Rightarrow v_j = \sum_{i=1}^{s} \mathrm{wt}(q_i)$.
- Edges in the subdivision of P: (v, v') iff $\overline{C_v} \cap \overline{C_{v'}}$ are codimension 1. □

Example:



$p \rightsquigarrow q_1$, $\mathrm{wt}(q_1) = 1 \det|e_2, -e_1| = 1$

$\mathrm{wt}(q') = 1 \det|e_1, -e_2| = 1$ $\Rightarrow v = [1,1]$

p' gives $[2,0] = v'$ We get 

etc.

[See Lecture XXIII]